THE KLEIN-GORDON EQUATION IN THE ANTI-DE SITTER COSMOLOGY

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ABSTRACT. This paper deals with the Klein-Gordon equation on the Poincaré chart of the 5-dimensional Anti-de Sitter universe. When the mass μ is larger than $-\frac{1}{4}$, the Cauchy problem is well posed despite the loss of global hyperbolicity due to the time-like horizon. We express the finite energy solutions in the form of a continuous Kaluza-Klein tower and we deduce a uniform decay as $\mid t \mid^{-\frac{3}{2}}$. We investigate the case $\mu = \frac{\nu^2-1}{2}$, $\nu \in \mathbb{N}^*$, which encompasses the gravitational fluctuations, $\nu = 4$, and the electromagnetic waves, $\nu = 2$. The propagation of the wave front set shows that the horizon acts like a perfect mirror. We establish that the smooth solutions decay as $\mid t \mid^{-2-\sqrt{\mu+\frac{1}{4}}}$, and we get global L^p estimates of Strichartz type. When ν is even, there appears a lacuna and the equipartition of the energy occurs at finite time for the compactly supported initial data, although the Huygens principle fails. We adress the cosmological model of the negative tension Minkowski brane, on which a Robin boundary condition is imposed. We prove the hyperbolic mixed problem is well-posed and the normalizable solutions can be expanded into a discrete Kaluza-Klein tower. We establish some $L^2 - L^\infty$ estimates in suitable weighted Sobolev spaces.

I. INTRODUCTION

The Anti-de-Sitter space-time AdS^{n+1}, is the unique $n + 1$-dimensional maximally symmetric solution without singularity, of the Einstein equations in the vacuum, with a negative cosmological constant (see the Appendix for a brief presentation). The recent surge of interest about AdS^5 came from the hope to construct a quantum gravity from a string theory. This dream has aroused a huge literature by the physicists, but there are few mathematical works dealing with the PDE of the fields theory in this geometrical framework. The problems arising in the context of AdS^{n+1} are somewhat unusual because of the crucial property of this lorentzian manifold : the loss of global hyperbolicity due to the existence of a time-like horizon. There are also closed time-like curves, but this unpleasant property disappears when we consider the universal covering $CAdS^{n+1}$, hence the main issue consists in understanding the role of the horizon in the existence, the uniqueness, and the qualitative properties of the solutions of the wave equations. We have investigated the Dirac system in $CAdS^4$ in [4] and the wave equations associated to each spherical harmonics of the scalar/electromagnetic/gravitational fluctuations on the whole $CAdS^{n+1}$ has been studied by A. Ishibashi and R. M. Wald [15]. The scalar waves on $CAdS^4$ were discussed in [3] and [8].

In this paper we consider the Klein-Gordon equation in the Poincaré patch of AdS^5. It is a strictly included subdomain $\mathcal{P}$ of AdS^5 that plays a fundamental role in brane cosmology (see e.g. [19]). This manifold is defined by

$$(I.1) \qquad \mathcal{P} := \mathbb{R}_t \times \mathbb{R}^3_{\mathbf{x}} \times]0, \infty[_z, \;\; g_{\mu\nu} d^\mu dx^\nu = \left(\frac{1}{kz}\right)^2 \left(dt^2 - d\mathbf{x}^2 - dz^2\right).$$

The boundary of this universe, that is located at $z = 0$, is time-like and we can see that many null geodesics hit this horizon, so $\mathcal{P}$ is not globally hyperbolic. The main aim of this paper consists in the understanding of the role of this horizon related to the propagation of the fields and their asymptotic behaviours. An important motivation for our work is the study of the gravitational

waves in this geometry that are some fluctuations u_{ij} of the metric AdS^5

$$ds^2 = \left(\frac{1}{kz}\right)^2 \left(dt^2 - (\delta_{ij} + u_{ij})dx^i dx^j - dz^2\right).$$

They are solutions of the linearized Einstein equations, which are simply reduced to the D'Alembertian in AdS^5

$$\Box_g u = 0, \quad \Box_g := \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\mu} \left(\sqrt{|g|} g^{\mu\nu} \frac{\partial}{\partial x^\nu}\right).$$

More generally, we consider the Klein-Gordon equation

(I.2) $$\Box_g u + \lambda k^2 u = 0,$$

with $\lambda \in \mathbb{R}$. In fact this equation is a master equation that appears for other fields : for the scalar fields, λ is the mass, for the vector electromagnetic fields $\lambda = -3$. If we put $\Phi =: z^{-\frac{3}{2}} u$ and $\mu := \frac{15}{4} + \lambda$, the equation (I.2) on $\mathcal{P}$ takes the very simple form of the free wave equation on the 1+4-dimensional half Minkowski space-time $\mathbb{R}_t \times \mathbb{R}^3_x \times]0, \infty[_z$, pertubed by a singular cartesian potential $\frac{\mu}{z^2}$:

(I.3) $$\left(\partial_t^2 - \Delta_\mathbf{x} - \partial_z^2 + \frac{\mu}{z^2}\right) \Phi = 0.$$

Since $\mathcal{P}$ is not globally hyperbolic, the well-posedness of the Cauchy problem is doubtful and the question arises of the necessity to impose some boundary condition on the time-like horizon. Nevertheless, we can establish that no time-like geodesic hits this boundary, therefore we may hope that the Cauchy problem is well-posed for the wave equations when the mass of the field is large enough. In the following part, we show that this is indeed the case when $\mu > -\frac{1}{4}$ (i.e. $\lambda > -4$), and there exists a unique solution of the initial value problem when the natural energy associated with (I.3) is finite. In fact this constraint is equivalent to the Dirichlet condition on the horizon. We also prove that the solutions are a superposition of a continuum Klein-Gordon fields (the so called Kaluza-Klein tower). We deduce that the smooth solutions decay uniformly as $|t|^{-\frac{3}{2}}$ and behave near the horizon as $z^{-\frac{1}{2}-\sqrt{\mu+\frac{1}{4}}}$.

In part 3, we obtain more precise properties when the mass μ has the form $\mu = \frac{\nu^2-1}{4}$, $\nu \in \mathbb{N}^*$. In this case (I.3) is closely linked with the wave equation in a higher dimension Minkowski space. This spectrum of mass includes the gravitational waves ($\mu = \frac{15}{4}$), and the electromagnetic waves ($\mu = \frac{3}{4}$). The Huygens principle fails for (I.3), but a lacuna appears, and the equipartition of the energy occurs at finite time for the compactly supported data, when ν is even. Moreover for all ν, the horizon acts like a perfect miror : the singularities are reflected according to the Descartes law. We prove a strong decay of the smooth solutions that behave as $|t|^{-2-\sqrt{\mu+\frac{1}{4}}} z^{-\frac{1}{2}-\sqrt{\mu+\frac{1}{4}}}$ We establish also global L^p space-time estimates of Strichartz type in weighted spaces for the finite energy solutions.

In the fourth part, we adress the cosmological problem of the Minkowski brane (see e.g. [19]). This brane is just the submanifold $\mathbb{R}_t \times \mathbb{R}^3_x \times \{z = 1\}$. In [5] we have developed a complete analysis of the gravitational fluctuations for the positive-tension Minkowski brane that is the boundary of $\mathbb{R}_t \times \mathbb{R}^3_x \times]1, \infty[_z$. In this paper, we investigate the Klein-Gordon equation for the negative-tension Minkowski brane considered as the part $z = 1$ of the boundary of the Anti-de Sitter bulk $\mathcal{B} := \mathbb{R}_t \times \mathbb{R}^3_x \times]0, 1[_z$. The dynamics of the field is given by the equation (I.3) in $\mathcal{B}$ and the Neumann condition on the brane, $\partial_z u = 0$, that is equivalent to

(I.4) $$\partial_z \Phi(t, \mathbf{x}, 1) + \frac{3}{2} \Phi(t, \mathbf{x}, 1) = 0.$$

We solve the mixed problem and expand the normalizable solutions in the form of a discrete Kaluza-Klein tower ; this result provides a rigorous functional framework to the expansions of the physicists.

We establish some $L^2 - L^\infty$ estimates that express that the smooth fields decay as $t^{-\frac{3}{2}} z^{-\frac{1}{2} - \sqrt{\mu + \frac{1}{4}}}$ again. Finally a short appendix presents the basics elements of the AdS^5 brane cosmology.

II. FINITE ENERGY SOLUTIONS ON AdS^5

In this part we investigate the finite energy solutions of the Klein-Gordon equation (I.3) where μ is a real number satisfying

$$\text{(II.1)} \qquad -\frac{1}{4} < \mu.$$

We shall see that this constraint is natural to assure the positivity of the energy that is the formally conserved quantity associated with the equation (I.3) :

$$\text{(II.2)} \qquad E(\Phi, t) := \int_{\mathbb{R}^3} \int_0^\infty \mid \nabla_{t,\mathbf{x},z} \Phi(t, \mathbf{x}, z) \mid^2 + \frac{\mu}{z^2} \mid \Phi(t, \mathbf{x}, z) \mid^2 d\mathbf{x} dz$$

The solutions Φ of (I.3) such that for all time t

$$\text{(II.3)} \qquad 0 \leq E(\Phi, t) < \infty$$

are called *finite energy solutions*. We prove that under this constraint, *i.e.* in a suitable functional framework, there is a unique dynamics associated to equation (I.3).

To ensure that the energy (II.2) is finite, it is natural to impose that for all fixed time t, $\Phi(t, .)$ belongs to $H^1(\mathbb{R}^3_{\mathbf{x}} \times]0, 1[_z)$. Since $\Phi(t, .) \in C^0([0, 1]_z; H^{\frac{1}{2}}(\mathbb{R}^3_{\mathbf{x}}))$, the constraint $\frac{1}{z} \Phi(t, .) \in L^2(\mathbb{R}^3_{\mathbf{x}} \times]0, 1[_z)$ implies that $\Phi(t, \mathbf{x}, 0) = 0$. Hence the finiteness of the energy implicitly requires the Dirichlet condition on the time-like horizon $z = 0$ of the anti-de Sitter space-time. Now we construct the functional framework associated to the energy. Following J. Deny and J-L. Lions [10], given an open set $\Omega \subset \mathbb{R}^N_{\mathbf{z}}$ and a Hilbert space X, we introduce the Beppo Levi space $BL_0^1(\Omega; X)$ (respectively $BL^1(\Omega; X)$) as the completion of the space of the test functions $C_0^\infty(\Omega; X)$ (respectively $C_0^\infty(\overline{\Omega}; X)$) for the norm $\|\nabla.\|_{L^2(\Omega;X)}$ (and we omit X when $X = \mathbb{C}$). When $N \geq 3$ these spaces are sets of distributions and we have

$$\text{(II.4)} \qquad \int_\Omega \frac{\mid \phi(\mathbf{z}) \mid^2_X}{\mid \mathbf{z} \mid^2} d\mathbf{z} \leq \frac{4}{(N-2)^2} \int_\Omega \mid \nabla_{\mathbf{z}} \phi(\mathbf{z}) \mid^2_X d\mathbf{z}$$

and also

$$\text{(II.5)} \qquad BL_0^1(\Omega) \subset L^{\frac{2N}{N-2}}(\Omega, d\mathbf{z}).$$

Thanks to the Hardy inequality

$$\text{(II.6)} \qquad \forall \phi \in C_0^\infty(]0, \infty[; X), \ \ \forall A > 0, \ \ \int_0^A \frac{1}{z^2} \left| \int_0^z \phi(\zeta) d\zeta \right|^2_X dz \leq 4 \int_0^A |\phi(z)|^2_X dz$$

we see that for all $\phi \in BL_0^1(\mathbb{R}^3_{\mathbf{x}} \times]0, \infty[_z)$, we have :

$$\text{(II.7)} \qquad \int_{\mathbb{R}^3} \int_0^\infty \mid \partial_z \phi(\mathbf{x}, z) \mid^2 + \frac{\mu}{z^2} \mid \phi(\mathbf{x}, z) \mid^2 d\mathbf{x} dz \leq (1 + 4 \mid \mu \mid) \int_{\mathbb{R}^3} \int_0^\infty \mid \partial_z \phi(\mathbf{x}, z) \mid^2 d\mathbf{x} dz$$

and when $-\frac{1}{4} < \mu \leq 0$

$$\text{(II.8)} \qquad (1 + 4\mu) \int_{\mathbb{R}^3} \int_0^\infty \mid \partial_z \phi(\mathbf{x}, z) \mid^2 d\mathbf{x} dz \leq \int_{\mathbb{R}^3} \int_0^\infty \mid \partial_z \phi(\mathbf{x}, z) \mid^2 + \frac{\mu}{z^2} \mid \phi(\mathbf{x}, z) \mid^2 d\mathbf{x} dz$$

hence (II.3) is satisfied by the solutions Φ of (I.3) such that

$$\text{(II.9)} \qquad \Phi \in C^0 \left(\mathbb{R}_t; BL_0^1 \left(\mathbb{R}^3_{\mathbf{x}} \times]0, \infty[_z \right) \right), \ \ \partial_t \Phi \in C^0 \left(\mathbb{R}_t; L^2 \left(\mathbb{R}^3_{\mathbf{x}} \times]0, \infty[_z \right) \right).$$

In the sequel, $BL_0^1 \left(\mathbb{R}^3_{\mathbf{x}} \times]0, \infty[_z \right)$ is endowed with the norm

$$\text{(II.10)} \qquad \|\Phi\|^2_{BL_0^1} := \int_{\mathbb{R}^3} \int_0^\infty \mid \nabla_{\mathbf{x},z} \Phi(\mathbf{x}, z) \mid^2 + \frac{\mu}{z^2} \mid \Phi(\mathbf{x}, z) \mid^2 d\mathbf{x} dz.$$

It will be useful to note that

$$(II.11) \qquad \|\Phi\|_{BL_0^1}^2 = \int_{\mathbb{R}^3} \int_0^\infty | \nabla_{\mathbf{x}} \Phi(\mathbf{x}, z) |^2 + \left| \partial_z \Phi(\mathbf{x}, z) + \frac{\alpha_\pm}{z} \Phi(\mathbf{x}, z) \right|^2 d\mathbf{x} dz,$$

with

$$(II.12) \qquad \alpha_\pm = -\frac{1}{2} \pm \sqrt{\mu + \frac{1}{4}}.$$

We are ready to state the result of existence of the finite energy solutions.

Theorem II.1. *For any $\mu > -\frac{1}{4}$, given $\Phi_0 \in BL_0^1 \left(\mathbb{R}_{\mathbf{x}}^3 \times]0, \infty[_z \right)$ and $\Phi_1 \in L^2 \left(\mathbb{R}_{\mathbf{x}}^3 \times]0, \infty[_z \right)$, there exists a unique solution Φ of (I.3) satisfying (II.9) and the Cauchy condition*

$$(II.13) \qquad \Phi(0, \mathbf{x}, z) = \Phi_0(\mathbf{x}, z), \quad \partial_t \Phi(0, \mathbf{x}, z)) = \Phi_1(\mathbf{x}, z), \quad (\mathbf{x}, z) \in \mathbb{R}^3 \times]0, \infty[.$$

Moreover the energy (II.2) is conserved :

$$(II.14) \qquad \forall t \in \mathbb{R}, \quad E(\Phi, t) = E(\Phi, 0),$$

and if $\Phi_0(\mathbf{x}, z) = \Phi_1(\mathbf{x}, z) = 0$ when $| \mathbf{x} | \geq R$ or $| z | \geq R$, then $\Phi(t, \mathbf{x}, z) = 0$ when $| \mathbf{x} | \geq R + | t |$ or $| z | \geq R + | t |$.

Proof of Theorem II.1. We introduce the densely defined operator on $BL_0^1 \left(\mathbb{R}_{\mathbf{x}}^3 \times]0, \infty[_z \right) \times L^2 \left(\mathbb{R}_{\mathbf{x}}^3 \times]0, \infty[_z \right)$, given by

$$A_0 := \frac{1}{i} \begin{pmatrix} 0 & 1 \\ \Delta_{\mathbf{x}} + \partial_z^2 - \frac{\mu}{z^2} & 0 \end{pmatrix}, \quad Dom(A_0) = C_0^\infty \left(\mathbb{R}_{\mathbf{x}}^3 \times]0, \infty[_z \right) \times C_0^\infty \left(\mathbb{R}_{\mathbf{x}}^3 \times]0, \infty[_z \right).$$

A_0 is obviously symmetric. Now we prove that it is essentially self-adjoint. We easily show that its adjoint A_0^* is defined by

$$A_0^* := \frac{1}{i} \begin{pmatrix} 0 & 1 \\ \Delta_{\mathbf{x}} + \partial_z^2 - \frac{\mu}{z^2} & 0 \end{pmatrix}, \quad Dom(A_0^*) = \left\{ \Phi_0 \in BL_0^1; \ -\Delta_{\mathbf{x},z} \Phi_0 + \frac{\mu}{z^2} \Phi_0 \in L^2 \right\} \times H_0^1,$$

where $H_0^1 = BL_0^1 \cap L^2$ is the usual Sobolev space. We consider $(\Phi_0^\pm, \Phi_1^\pm) \in Ker(A_0^* \pm i)$. Then $\Phi_1^\pm = \pm \Phi_0^\pm \in H_0^1$ and

$$\left[-\Delta_{\mathbf{x},z} + \frac{\mu}{z^2} + 1 \right] \Phi_0^\pm = 0.$$

Since $\phi \mapsto \Delta_{\mathbf{x},z} \phi$ and $\phi \mapsto \frac{1}{z^2} \phi$ are bounded from $H_0^1 \left(\mathbb{R}_{\mathbf{x}}^3 \times]0, \infty[_z \right)$ to its dual space $H^{-1} \left(\mathbb{R}_{\mathbf{x}}^3 \times]0, \infty[_z \right)$, we deduce that

$$\|\Phi_0^\pm\|_{BL_0^1}^2 + \|\Phi_0^\pm\|_{L^2}^2 = \left\langle \left[-\Delta_{\mathbf{x},z} + \frac{\mu}{z^2} + 1 \right] \Phi_0^\pm; \Phi_0^\pm \right\rangle_{H^{-1}, H_0^1} = 0,$$

hence we conclude that $\Phi_j^\pm = 0$ and A_0 is essentially self-adjoint. If A is its unique self-adjoint extension, $\Phi(t, .)$ given by $(\Phi(t), \partial_t \Phi(t)) = e^{itA}(\Phi_0, \Phi_1)$ satisfies (I.3), (II.9), (II.13) and (II.14). The uniqueness is established by a classical way. Given a solution Φ of (I.3), (II.9), (II.13), we put for all $\varepsilon > 0$, $\Phi_\varepsilon(t) = \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \Phi(\tau) d\tau$. Then $\Phi_\varepsilon \in C^1(\mathbb{R}_t; BL_0^1)$ and $\partial_t \Phi_\varepsilon \in C^1(\mathbb{R}_t; L^2)$. Then $t \mapsto E(\Phi_\varepsilon, t)$ is C^1 and we have

$$\frac{d}{dt} E(\Phi_\varepsilon, t) = 2\Re \left\langle \left[\partial_t^2 - \Delta_{\mathbf{x},z} + \frac{\mu}{z^2} \right] \Phi_\varepsilon; \overline{\partial_t \Phi_\varepsilon} \right\rangle_{H^{-1}, H_0^1} = 0.$$

We get $E(\Phi_\varepsilon, t) = E(\Phi_\varepsilon, 0)$. Since $\Phi_\varepsilon \to \Phi$ in $C^0(\mathbb{R}_t; BL_0^1)$ and $\partial_t \Phi_\varepsilon \to \partial_t \Phi$ in $C^0(\mathbb{R}_t; L^2)$ as $\varepsilon \to 0$, we conclude that (II.14) is satisfied by Φ. Finally we establish the result of finite velocity propagation by the usual way. For a sake of simplicity, we assume the field is real valued. Given

$R_1,\ R_2 > 0,\ T > 0,\ \mathbf{x}_0 \in \mathbb{R}^3$, and $z_0 \in \mathbb{R}$ such that $\mid \mathbf{x}_0 \mid > R_1 + T,\ \mid z_0 \mid > R_2 + T$, we integrate the Pointing vector

$$\overrightarrow{p}(t,\mathbf{x},z) := \begin{pmatrix} \mid \nabla_{t,\mathbf{x}}\Phi(t,\mathbf{x},z) \mid^2 + \mid \partial_z \Phi(t,\mathbf{x},z) + \frac{\alpha_\pm}{z}\Phi(t,\mathbf{x},z) \mid^2 \\ -2\partial_t \Phi(t,\mathbf{x},z)\nabla_{\mathbf{x}}\Phi(t,\mathbf{x},z) \\ -2\partial_t \Phi(t,\mathbf{x},z)\left(\partial_z \Phi(t,\mathbf{x},z) + \frac{\alpha_\pm}{z}\Phi(t,\mathbf{x},z)\right) \end{pmatrix}$$

on the domain $\{(t,\mathbf{x},z);\ 0 \le t \le T,\ \mid \mathbf{x} - \mathbf{x}_0 \mid \le R_1 + T - t,\ \mid z - z_0 \mid \le R_2 + T - t\}$. Since $\nabla_{t,\mathbf{x},z} \cdot \overrightarrow{p} = 0$, we get the control of the local energy :

$$\int_{|\mathbf{x}-\mathbf{x}_0| \le R_1} \int_{|z-z_0| \le R_2} \mid \nabla_{t,\mathbf{x}}\Phi(T,\mathbf{x},z) \mid^2 + \left| \partial_z \Phi(T,\mathbf{x},z) + \frac{\alpha_\pm}{z}\Phi(T,\mathbf{x},z) \right|^2 d\mathbf{x}dz$$

$$\le \int_{|\mathbf{x}-\mathbf{x}_0| \le R_1+T} \int_{|z-z_0| \le R_2+T} \mid \nabla_{t,\mathbf{x}}\Phi(0,\mathbf{x},z) \mid^2 + \left| \partial_z \Phi(0,\mathbf{x},z) + \frac{\alpha_\pm}{z}\Phi(0,\mathbf{x},z) \right|^2 d\mathbf{x}dz,$$

and the result of finite velocity propagation is a straightforward consequence of this estimate.

$$Q.E.D.$$

In brane cosmology it is important to express the fields propagating in the Anti-de Sitter universe, as a superposition of particular solutions with an infinite energy, called the Kaluza-Klein tower, by decoupling the variables (t,x) and z, the space variable of depth.

Theorem II.2. *For any $\Phi_0 \in BL_0^1(\mathbb{R}^3 \times]0,\infty[)$, $\Phi_1 \in L^2(\mathbb{R}^3 \times]0,\infty[)$, the solution Φ of (I.3), (II.9) and (II.13) can be expressed as*

(II.15)
$$\Phi(t,\mathbf{x},z) = \lim_{M\to\infty} \int_0^M \phi_m(t,\mathbf{x})\sqrt{mz}J_\lambda(mz)dm \quad in \quad C^0\left(\mathbb{R}_t; BL_0^1\left(\mathbb{R}^3 \times]0,\infty[\right)\right),$$
$$\partial_t \Phi(t,\mathbf{x},z) = \lim_{M\to\infty} \int_0^M \partial_t \phi_m(t,\mathbf{x})\sqrt{mz}J_\lambda(mz)dm \quad in \quad C^0\left(\mathbb{R}_t; L^2\left(\mathbb{R}^3 \times]0,\infty[\right)\right),$$

where

(II.16)
$$\lambda := \sqrt{\mu + \frac{1}{4}},$$

and for any $T > 0$

(II.17)
$$\phi_m \in L^2\left(]0,\infty[_m; C^0\left([-T,T]_t; BL^1\left(\mathbb{R}_{\mathbf{x}}^3\right)\right)\right) \cap L_{loc}^2\left(]0,\infty[_m; C^0\left([-T,T]_t; H^1\left(\mathbb{R}_{\mathbf{x}}^3\right)\right)\right),$$
$$\partial_t \phi_m \in L^2\left(]0,\infty[_m; C^0\left([-T,T]_t; L^2\left(\mathbb{R}_{\mathbf{x}}^3\right)\right)\right)$$

is solution for almost all $m > 0$, of

(II.18)
$$\partial_t^2 \phi_m - \Delta_{\mathbf{x}}\phi_m + m^2\phi_m = 0,\ \ t \in \mathbb{R},\ \ \mathbf{x} \in \mathbb{R}^3.$$

Moreover,

(II.19)
$$\| \Phi_0 \|_{BL_0^1}^2 + \| \Phi_1 \|_{L^2}^2 = \int_0^\infty \| \nabla_{t,\mathbf{x}}\phi_m(t) \|_{L^2(\mathbb{R}^3)}^2 + m^2 \| \phi_m(t) \|_{L^2(\mathbb{R}^3)}^2\ dm.$$

Proof of Theorem II.2. We shall use some results on the Sturm-Liouville theory (see *e.g.* [20], [23], [24]). Given $\mu > -\frac{1}{4}$, we consider the Bessel operator

(II.20)
$$\mathrm{P}_\mu := -\frac{d^2}{dz^2} + \frac{\mu}{z^2},$$

and we put

(II.21)
$$\mathfrak{D}(\mu) := \left\{ u \in L^2(]0,\infty[);\ \mathrm{P}_\mu u \in L^2(]0,\infty[) \right\}.$$

We introduce the densely defined operator $\mathbf{h}_\mu$ on $L^2(]0,\infty[)$ defined by

(II.22) $\qquad Dom(\mathbf{h}_\mu) = \mathfrak{D}_0(\mu) := \mathfrak{D}(\mu) \cap H_0^1(]0,\infty[), \quad \forall u \in Dom(\mathbf{h}_\mu) \ \ \mathbf{h}_\mu u = \mathrm{P}_\mu u.$

The Hardy inequality assures that

(II.23) $\qquad\qquad\qquad\qquad \mathfrak{D}_0(\mu) = \left\{ u \in \mathfrak{D}(\mu); \ u', \ z^{-1}u \in L^2(]0,\infty[) \right\}.$

It is easy to prove that $\mathbf{h}_\mu$ is a positive symmetric operator on $L^2(]0,\infty[)$ and the variational method shows that $Ran(\mathbf{h}_\mu + 1) = L^2(]0,\infty[)$, hence $\mathbf{h}_\mu$ is self-adjoint. In fact, according to [16], [21], $\mathbf{h}_\mu$ is just the Friedrichs extension of the differential operator P_μ and

(II.24) $\qquad\qquad\qquad \forall u \in \mathfrak{D}_0(\mu), \ \ \lim_{z \to 0^+} z^{-\frac{1}{2}+\lambda} u(z) = \lim_{z \to 0^+} z^{\frac{1}{2}+\lambda} u'(z) = 0.$

When $\mu \geq \frac{3}{2}$ (i.e. $\lambda \geq 1$), P_μ is essentially self-adjoint on $C_0^\infty(]0,\infty[)$ and $\mathfrak{D}_0(\mu) = \mathfrak{D}(\mu)$. When $\mu \in]-\frac{1}{4}, \frac{3}{4}[$ (i.e. $\lambda \in]0,1[$), P_μ is in the limit-circle case at the origin, singular for $\mu \neq 0$, and all the self-adjoint extensions $\mathbf{h}^\omega$, are characterized by a boundary condition at $z = 0$ associated to any $\omega \in \mathfrak{D} \setminus \mathfrak{D}_0 : Dom(\mathbf{h}^\omega) = \left\{ u \in \mathfrak{D}; \lim_{z \to 0^+} u'\overline{\omega} - u\overline{\omega'} = 0 \right\}$. As regards the Friedrichs extension $\mathbf{h}_\mu$, the following sharpened asymptotics are established in [11] for $u \in Dom(\mathbf{h}_\mu)$:

(II.25) $\qquad\qquad \lim_{z \to 0^+} z^{-\frac{1}{2}-\lambda} u(z) = K_\lambda(u) := u(1) + \int_0^1 t^{-2\lambda-1} \left(\int_0^t z^{\lambda+\frac{1}{2}} \mathrm{P}_\mu u(z) dz \right) dt,$

(II.26) $\qquad\qquad\qquad \lim_{z \to 0^+} z^{\frac{1}{2}-\lambda} u'(z) = \left(\lambda + \frac{1}{2} \right) K_\lambda(u), \quad \lim_{z \to 0^+} u(z)u'(z) = 0.$

The spectral representation is given by the Hankel transform of $u \in L^2(]0,\infty[)$,

(II.27) $\qquad \mathrm{H}_\lambda u(m) := \lim_{R \to \infty} \hat{H}_\lambda^R u \ \ in \ \ L^2(]0,\infty[_m, dm), \ \ \hat{H}_\lambda^R u(m) := \int_0^R \sqrt{mz} J_\lambda(mz) u(z) dz.$

It is a well known result that for any $\lambda > 0$, H_λ is an involutive isometry from $L^2(]0,\infty[_z, dz)$ onto $L^2(]0,\infty[_m, dm)$ ([24], Theorem 129 with $\Re(s) = 2^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}\lambda + \frac{1}{2}s + \frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\lambda - \frac{1}{2}s + \frac{1}{4}\right)}, \lambda > -1)$:

(II.28) $\qquad u(z) = \lim_{R \to \infty} \check{H}_\lambda^R [\mathrm{H}_\lambda u] (z) \ \ in \ \ L^2(]0,\infty[_z, dz), \ \ \check{H}_\lambda^R v(z) := \int_0^R \sqrt{mz} J_\lambda(mz) v(m) dm.$

More generally, for any $u \in L^2(\mathbb{R}_\mathrm{x}^3 \times]0,\infty[_z)$ the Fubini theorem implies that

$$\| u - \check{H}_\lambda^R [\mathrm{H}_\lambda u] \|_{L^2(\mathbb{R}_\mathrm{x}^3 \times]0,\infty[_z)}^2 = \int_{\mathbb{R}^3} \| u(\mathrm{x},.) - \check{H}_\lambda^R [\mathrm{H}_\lambda u(\mathrm{x},.)] \|_{L^2(]0,\infty[_z)}^2 \, d\mathrm{x},$$

and since $\| u(\mathrm{x},.) - \check{H}_\lambda^R [\mathrm{H}_\lambda u(\mathrm{x},.)] \|_{L^2(]0,\infty[_z)}^2 \leq 4 \| u(\mathrm{x},.) \|_{L^2(]0,\infty[_z)}^2$, we deduce from (II.28) an the dominated convergence theorem that

$$u(\mathrm{x}, z) = \lim_{R \to \infty} \check{H}_\lambda^R [\mathrm{H}_\lambda u(\mathrm{x},.)] (z) \ \ in \ \ L^2\left(\mathbb{R}_\mathrm{x}^3 \times]0,\infty[_z\right).$$

Furthermore, since $\check{H}_\lambda^R$ and H_λ are isometric, this limit is uniform on the compacts of $L^2\left(\mathbb{R}_\mathrm{x}^3 \times]0,\infty[_z\right)$, hence we conclude that for any $u \in C^0\left(\mathbb{R}_t; L^2\left(\mathbb{R}_\mathrm{x}^3 \times]0,\infty[_z\right)\right)$ we have

(II.29) $\qquad u(t, \mathrm{x}, z) = \lim_{R \to \infty} \check{H}_\lambda^R [\mathrm{H}_\lambda u(t, \mathrm{x},.)] (z) \ \ in \ \ C^0\left(\mathbb{R}_t; L^2\left(\mathbb{R}_\mathrm{x}^3 \times]0,\infty[_z\right)\right).$

Finally for $u \in Dom(\mathbf{h}_\mu)$ we have for all $m > 0$

(II.30) $\qquad\qquad\qquad\qquad\qquad \mathrm{H}_\lambda \mathbf{h}_\mu u(m) = m^2 \mathrm{H}_\lambda u(m).$

This property is easily obtained with an integration by part by using the asymptotic behaviours at the origin (II.25), (II.26). and the equality

(II.31) $\qquad\qquad\qquad \mathrm{P}_\mu \left(\sqrt{mz} J_\lambda(mz) \right) = m^2 \sqrt{mz} J_\lambda(mz), \ \ 0 < m, \ \ 0 < z.$

We are now ready to apply these properties to prove the theorem. Given $t \in \mathbb{R}$, for almost all $\mathbf{x} \in \mathbb{R}^3$, the map $z \mapsto \nabla_{\mathbf{x}} \Phi(t, \mathbf{x}, z)$ belongs to $L^2(\mathbb{R}_z^+)$. Thus for almost all $m > 0$ we can introduce

$$\phi_m(t, \mathbf{x}) := \mathrm{H}_\lambda \Phi(t, \mathbf{x}, .)(m),$$

that belongs to $L^2\left(\mathbb{R}_m^+; BL^1(\mathbb{R}_{\mathbf{x}}^3)\right)$. Moreover, since $\Phi \in C^0\left(\mathbb{R}_t; L^2\left(\mathbb{R}_z^+; BL^1\left(\mathbb{R}_{\mathbf{x}}^3\right)\right)\right)$ with $\partial_t \Phi \in C^0\left(\mathbb{R}_t; L^2\left(]0, \infty[_z \times \mathbb{R}^3\right)\right)$, then

(II.32)
$$\phi_m \in C^0\left(\mathbb{R}_t; L^2\left(\mathbb{R}_m^+; BL^1\left(\mathbb{R}_{\mathbf{x}}^3\right)\right)\right), \quad \partial_t \phi_m \in C^0\left(\mathbb{R}_t; L^2\left(\mathbb{R}_m^+; L^2\left(\mathbb{R}_{\mathbf{x}}^3\right)\right)\right),$$

therefore (II.15) is established. We also have :

(II.33)
$$\begin{cases} \phi_m(0, \mathbf{x}) = \mathrm{H}_\lambda\left(\Phi_0(\mathbf{x}, .)\right)(m) \in L^2\left(\mathbb{R}_m^+; BL^1\left(\mathbb{R}_{\mathbf{x}}^3\right)\right), \\ \partial_t \phi_m(0, \mathbf{x}) = \mathrm{H}_\lambda\left(\Phi_1(\mathbf{x}, .)\right)(m) \in L^2\left(\mathbb{R}_m^+; L^2\left(\mathbb{R}_{\mathbf{x}}^3\right)\right). \end{cases}$$

Moreover we have

$$\nabla_{t,\mathbf{x}} \Phi(t, \mathbf{x}, z) = \mathrm{H}_\lambda\left[\nabla_{t,\mathbf{x}} \phi_m(t, \mathbf{x})\right](z) \in C^0\left(\mathbb{R}_t; L^2(\mathbb{R}_{\mathbf{x}}^3 \times]0, \infty[_z)\right),$$

hence

(II.34)
$$\int_0^\infty \int_{\mathbb{R}_{\mathbf{x}}^3} |\nabla_{t,\mathbf{x}} \Phi(t, \mathbf{x}, z)|^2 \, dz d\mathbf{x} = \int_0^\infty \int_{\mathbb{R}_{\mathbf{x}}^3} |\nabla_{t,\mathbf{x}} \phi_m(t, \mathbf{x}, m)|^2 \, dm d\mathbf{x}.$$

We remark that for $u \in \mathfrak{D}(\mathbf{h}_\mu)$ we have

$$\int_0^\infty \left| u'(z) + \frac{\alpha_\pm}{z} u(z) \right|^2 dz = <\mathbf{h}_\mu u, u>_{L^2(\mathbb{R}^+)} = \int_0^\infty |\mathrm{H}_\lambda u(m)|^2 m^2 dm.$$

This equality can be extended by density into an isometry from the closure of $\mathfrak{D}(\mathbf{h}_\mu)$ for the norm associated to the first integral, onto $L^2(\mathbb{R}_m^+, m^2 dm)$. We deduce that $m\phi_m \in C^0\left(\mathbb{R}_t; L^2\left(\mathbb{R}_m^+; L^2(\mathbb{R}_{\mathbf{x}}^3)\right)\right)$ hence with (II.32)

(II.35)
$$\forall a > 0, \quad \phi_m \in C^0\left(\mathbb{R}_t; L^2\left([a, \infty[_m; H^1\left(\mathbb{R}_{\mathbf{x}}^3\right)\right)\right),$$

and

(II.36)
$$\int_0^\infty \int_{\mathbb{R}_{\mathbf{x}}^3} \left| \partial_z \Phi(t, \mathbf{x}, z) + \frac{\alpha_\pm}{z} \Phi(t, \mathbf{x}, z) \right|^2 dz d\mathbf{x} = \int_0^\infty \int_{\mathbb{R}_{\mathbf{x}}^3} m^2 |\phi_m(t, \mathbf{x}, m)|^2 \, dm d\mathbf{x}.$$

Now (II.19) follows from (II.34) and (II.36).

Now we establish that ϕ_m is a finite energy solution of the Klein-Gordon equation for almost all $m > 0$. Thanks to (II.19), we see that the map $(\Phi_0, \Phi_1) \mapsto (\phi_m, \partial_t \phi_m)$ is continuous from $BL_0^1(\mathbb{R}^3 \times]0, \infty[) \times L^2(\mathbb{R}^3 \times]0, \infty[)$ to $C^0\left(\mathbb{R}_t; L^2\left(\mathbb{R}_m^+; BL^1(\mathbb{R}_{\mathbf{x}}^3)\right)\right) \times C^0\left(\mathbb{R}_t; L^2\left(\mathbb{R}_m^+; L^2(\mathbb{R}_{\mathbf{x}}^3)\right)\right)$, hence it is sufficient to prove that

(II.37)
$$(\partial_t^2 - \Delta_{\mathbf{x}} + m^2)\phi_m = 0 \;\; in \;\; \mathcal{D}'\left(\mathbb{R}_t \times \mathbb{R}_{\mathbf{x}}^3 \times]0, \infty[_m\right)$$

for a dense set of initial data. We choose $\Phi_0, \Phi_1 \in C_0^\infty(\mathbb{R}^3 \times]0, \infty[)$. Then the solution Φ of the Cauchy problem is compactly supported in space at each time and since $\Delta_{\mathbf{x}}$ and $-\Delta_{\mathbf{x},z} + \frac{\mu}{z^2}$ are commuting, $\Delta_{\mathbf{x}} \Phi$ is also a finite energy solution, and we have :

$$\Phi, \; \Delta_{\mathbf{x}} \Phi \in C^2\left(\mathbb{R}_t; L^2\left(\mathbb{R}^3 \times]0, \infty[\right)\right) \cap C^1\left(\mathbb{R}_t; H_0^1\left(\mathbb{R}^3 \times]0, \infty[\right)\right).$$

This implies that

$$\left[-\partial_z^2 + \frac{\mu}{z^2}\right] \Phi \in C^0\left(\mathbb{R}_t; L^2(\mathbb{R}^3 \times]0, \infty[)\right)$$

We deduce that

$$\Phi \in C^0\left(\mathbb{R}_t; L^2\left(\mathbb{R}_{\mathbf{x}}^3; \mathfrak{D}_0(\mu)\right)\right),$$

therefore ϕ_m belongs to $C^1\left(\mathbb{R}_t; L^2\left(\mathbb{R}_m^+; H^1(\mathbb{R}_\mathbf{x}^3)\right)\right) \times C^2\left(\mathbb{R}_t; L^2\left(\mathbb{R}_m^+; L^2(\mathbb{R}_\mathbf{x}^3)\right)\right)$, and

$$(\partial_t^2 - \Delta_\mathbf{x} + m^2)\phi_m = H_\lambda\left(\partial_t^2\Phi - \Delta_{\mathbf{x},z}\Phi - \frac{\mu}{z^2}\Phi\right) = 0.$$

It remains to prove that in the general case where $(\Phi_0, \Phi_1) \in BL_0^1\left(\mathbb{R}^3 \times]0,\infty[\right) \times L^2\left(\mathbb{R}^3 \times]0,\infty[\right)$, ϕ_m belongs to $C^0\left(\mathbb{R}_t; H^1\left(\mathbb{R}_\mathbf{x}^3\right)\right) \cap C^1\left(\mathbb{R}_t; L^2\left(\mathbb{R}_\mathbf{x}^3\right)\right)$ for almost all $m > 0$. We have established that for $0 < a$, $\phi_m \in C^0\left(\mathbb{R}_t; L^2\left([a,\infty[_m; H^1\left(\mathbb{R}_\mathbf{x}^3\right)\right)\right) \cap C^1\left(\mathbb{R}_t; L^2\left([a,\infty[_m; L^2\left(\mathbb{R}_\mathbf{x}^3\right)\right)\right)$ is solution of (II.33) and (II.37). We have proved in [5], p. 829-830, that this Cauchy problem is well posed in this functional framework and the solution belongs to $C^0\left(\mathbb{R}_t; H^1\left(\mathbb{R}_\mathbf{x}^3\right)\right) \cap C^1\left(\mathbb{R}_t; L^2\left(\mathbb{R}_\mathbf{x}^3\right)\right)$ for almost all $m > 0$.

$$Q.E.D.$$

Since the smooth solutions of the massive Klein-Gordon equation on the 3+1 dimensional Minkowski space-time decay as $\mid t \mid^{-\frac{3}{2}}$, we can use lemma 4.3 of [5] to obtain the same rate of decay uniformly for a suitable class of solutions of (I.3). The result above states a $L^1 - L^\infty$ estimate of von Wahl type in weighted spaces, in particular the factor $z^{-\lambda-\frac{1}{2}}$ in the uniform bound expresses that the horizon is repulsive. In the next part we establish some more strong properties for certain values of the mass, in particular for the gravitational or electromagnetic fluctuations.

Theorem II.3. *There exists $C > 0$ such that any finite energy solution Φ of (I.3), (II.13) satisfies the following estimate with $\lambda = \sqrt{\mu + \frac{1}{4}}$, provided the L^1 norms in the right member are finite :*

(II.38)
$$\left\| z^{-\lambda-\frac{1}{2}}\Phi(t,.) \right\|_{L^\infty(\mathbb{R}_\mathbf{x}^3 \times]0,\infty[_z)} \leq$$

$$C \mid t \mid^{-\frac{3}{2}} \sum_{j=0,1} \sum_{|\alpha|+j\leq 3} \|\partial_\mathbf{x}^\alpha \Phi_j\|_{L^1(\mathbb{R}_\mathbf{x}^3 \times]0,\infty[_z)} + \left\| \partial_\mathbf{x}^\alpha \left(-\partial_z^2 + \frac{\mu}{z^2}\right)^{\left[\frac{\lambda+3-|\alpha|-j}{2}\right]+1} \Phi_j \right\|_{L^1(\mathbb{R}_\mathbf{x}^3 \times]0,\infty[_z)}.$$

Proof of Theorem II.3. It sufficient to consider the case $\Phi_j \in C_0^\infty\left(\mathbb{R}_\mathbf{x}^3 \times]0,\infty[_z\right)$. Since the Bessel function satisfies $\mid J_\lambda(x) \mid \leq Cx^\lambda$, we can write

$$\int_0^M \left|\sqrt{mz}J_\lambda(mz)\phi_m(t,\mathbf{x})\right| dm \leq Cz^{\lambda+\frac{1}{2}}\int_0^M m^{\lambda+\frac{1}{2}} \parallel \phi_m(t,.) \parallel_{L^\infty(\mathbb{R}_\mathbf{x}^3)} dm.$$

We have

$$\phi_m(0,\mathbf{x}) = \phi_m^0(\mathbf{x}) := \int_0^\infty \sqrt{mz}J_\lambda(mz)\Phi_0(\mathbf{x},z)dz,$$

$$\partial_t\phi_m(0,\mathbf{x}) = \phi_m^1(\mathbf{x}) := \int_0^\infty \sqrt{mz}J_\lambda(mz)\Phi_1(\mathbf{x},z)dz.$$

We remark that for all $k \in \mathbb{N}$

$$\phi_m^j(\mathbf{x}) = m^{-2k}\int_0^\infty \sqrt{mz}J_\lambda(mz)\left(-\partial_z^2 + \frac{\mu}{z^2}\right)^k \Phi_j(\mathbf{x},z)dz.$$

Since $\sqrt{x}J_\lambda(x) \in L^\infty(]0,\infty[)$, we get

$$\parallel \partial_\mathbf{x}^\alpha \phi_m^j \parallel_{L^1(\mathbb{R}_\mathbf{x}^3)} \leq C(1+m)^{-2k} \parallel \Phi_j \parallel_{\alpha,k}$$

with

$$\parallel \Phi_j \parallel_{\alpha,k} := \|\partial_\mathbf{x}^\alpha \Phi_j\|_{L^1(\mathbb{R}_\mathbf{x}^3 \times]0,\infty[_z)} + \left\| \partial_\mathbf{x}^\alpha\left(-\partial_z^2 + \frac{\mu}{z^2}\right)^k \Phi_j \right\|_{L^1(\mathbb{R}_\mathbf{x}^3 \times]0,\infty[_z)}.$$

The equation (4.4) in [5] assures that

$$\parallel \phi_m(t,.) \parallel_{L^\infty(\mathbb{R}_\mathbf{x}^3)} \leq C(1+m\mid t \mid)^{-\frac{3}{2}} \sum_{j=0,1} \sum_{|\alpha|+j\leq 3} m^{3-|\alpha|-j} \parallel \partial_\mathbf{x}^\alpha \phi_m^j \parallel_{L^1(\mathbb{R}_\mathbf{x}^3)},$$

Then we get

$$\int_0^M |\sqrt{mz}J_\lambda(mz)\phi_m(t,\mathrm{x})|dm \leq$$

$$Cz^{\lambda+\frac{1}{2}} \sum_{j=0,1} \sum_{|\alpha|+j\leq 3} \int_0^M m^{\lambda+\frac{1}{2}+3-|\alpha|-j}(1+m)^{-2k}(1+m\mid t\mid)^{-\frac{3}{2}}dm \parallel \Phi_j \parallel_{\alpha,k}$$

$$Cz^{\lambda+\frac{1}{2}} \mid t \mid^{-\frac{3}{2}} \sum_{j=0,1} \sum_{|\alpha|+j\leq 3} \int_0^M m^{\lambda+2-|\alpha|-j}(1+m)^{-2k}dm \parallel \Phi_j \parallel_{\alpha,k}.$$

Since $\lambda > 0$, it is sufficient to choose $k = \left[\frac{\lambda+3-|\alpha|-j}{2}\right] + 1$ to assure that

$$\int_0^\infty m^{\lambda+2-|\alpha|-j}(1+m)^{-2k}dm < \infty.$$

We conclude that when the right member of (II.38) is finite, given $t \in \mathbb{R}$, the limit (II.15) holds in $L^\infty\left(\mathbb{R}_\mathrm{x}^3 \times]0,\infty[_z; z^{-\lambda-\frac{1}{2}}dxdz\right)$ and its norm is estimated by (II.38).

$$Q.E.D.$$

III. THE CASE OF THE MASS $\mu = \frac{\nu^2-1}{4}$, $\nu \in \mathbb{N}^*$.

The equations for the gravitational fluctuations ($\mu = \frac{15}{4}$) or the electromagnetic perturbations ($\mu = \frac{3}{4}$) belong to a large class for which $\mu = \frac{\nu^2-1}{4}$, $\nu \in \mathbb{N}^*$. For these values of μ, the finite energy solutions are closely linked to the finite energy solutions of the free wave equation on the Minkowski space-time with a higher dimension. The method rests on a very simple observation : we can consider the fifth space-like dimension $z > 0$, as the radial coordinate of some euclidean high-dimensional space $\mathbb{R}_\mathbf{z}^N$, $N \geq 2$, i.e. $z = \mid \mathbf{z} \mid$. We denote $Y_{l,m}$ the generalized spherical harmonics that form a orthonormal basis of $L^2(S^{N-1})$ of eigenfunctions of the Laplace-Beltrami operator satisfying $\Delta_{S^{N-1}}Y_{l,m} = -l(l+N-2)Y_{l,m}$. Here $l, m \in \mathbb{N}$ and m is bounded by the dimension of the space of harmonic homogeneous polynomials of degree l in N variables. Then it is straightforward to check that $\Phi \in L_{loc}^1(\mathbb{R}_t \times \mathbb{R}_\mathbf{x}^3 \times]0,\infty[_z)$ is solution of (I.3) with

$$(III.1) \qquad \mu = -\frac{1}{4} + \left(\frac{N}{2}+l-1\right)^2, \quad l, N \in \mathbb{N}, \quad 0 \leq l, \quad 2 \leq N,$$

if and only if

$$(III.2) \qquad \Psi(t,\mathbf{x},\mathbf{z}) := \mid \mathbf{z} \mid^{-\frac{N-1}{2}} \Phi(t,\mathbf{x},\mid \mathbf{z} \mid)Y_{l,m}\left(\frac{\mathbf{z}}{\mid \mathbf{z} \mid}\right)$$

is a solution in $L_{loc}^1\left(\mathbb{R}_t \times \mathbb{R}_\mathbf{x}^3 \times \left(\mathbb{R}_\mathbf{z}^N \setminus \{0\}\right)\right)$ of

$$(III.3) \qquad \left(\partial_t^2 - \Delta_\mathbf{x} - \Delta_\mathbf{z}\right)\Psi = 0, \quad (t,\mathbf{x},\mathbf{z}) \in \mathbb{R} \times \mathbb{R}^3 \times \left(\mathbb{R}^N \setminus \{0\}\right).$$

Now the crucial point is that when Φ is a finite energy solution in AdS^5, then Ψ is a finite energy free wave in the whole Minkowski space-time $\mathbb{R}_t \times \mathbb{R}_{\mathbf{x},N}^{3+N}$.

Lemma III.1. *Let Φ be satisfying (II.9), a solution of (I.3) where μ is given by (III.1) for some integers $N \geq 3$ and $l \geq 0$, or $N = 2$ and $l \geq 1$. Then Ψ defined by (III.2) satisfies*

$$(III.4) \qquad \Psi \in C^0\left(\mathbb{R}_t; BL^1\left(\mathbb{R}_{\mathbf{x},\mathbf{z}}^{3+N}\right)\right), \quad \partial_t\Psi \in C^0\left(\mathbb{R}_t; L^2\left(\mathbb{R}_{\mathbf{x},\mathbf{z}}^{3+N}\right)\right),$$

$$(III.5) \qquad \partial_t^2\Psi - \Delta_{\mathbf{x},\mathbf{z}}\Psi = 0, \quad (t,\mathbf{x},\mathbf{z}) \in \mathbb{R} \times \mathbb{R}^{3+N}.$$

Proof. We introduce the operator

$$\Pi_{l,m} : \; C_0^\infty\left(\mathbb{R}_{\mathbf{x}}^3\times]0,\infty[_z\right) \longrightarrow C_0^\infty\left(\mathbb{R}_{\mathbf{x}}^3\times\left(\mathbb{R}_{\mathbf{z}}^N\setminus\{0\}\right)\right),$$

(III.6)

$$\phi(\mathbf{x},z)\longmapsto (\Pi_{l,m}\phi)\,(\mathbf{x},\mathbf{z}) := \mid \mathbf{z}\mid^{-\frac{N-1}{2}}\phi(\mathbf{x},\mid \mathbf{z}\mid)Y_{l,m}\left(\frac{\mathbf{z}}{\mid \mathbf{z}\mid}\right).$$

It is clear that $\Pi_{l,m}$ can be extended in an isometry from $L^2\left(\mathbb{R}_{\mathbf{x}}^3\times]0,\infty[_z\right)$ to $L^2\left(\mathbb{R}_{\mathbf{x},\mathbf{z}}^{3+N}\right)$. Moreover we have

$$\Delta_{\mathbf{x},\mathbf{z}}\Pi_{l,m} = \Pi_{l,m}\left(\Delta_{\mathbf{x}} + \partial_z^2 - \frac{\mu}{z^2}\right),$$

therefore given $\phi,\;\phi'\in C_0^\infty\left(\mathbb{R}_{\mathbf{x}}^3\times]0,\infty[_z\right)$, we have

$$\left\langle \Pi_{l,m}\phi,\Pi_{l',m'}'\phi'\right\rangle_{BL^1(\mathbb{R}^{3+N})} = -<\Delta_{\mathbf{x},\mathbf{z}}\Pi_{l,m}\phi,\Pi_{l',m'}\phi'>_{L^2(\mathbb{R}^{3+N})}$$

(III.7)

$$= -<\Pi_{l,m}\left(\Delta_{\mathbf{x}}+\partial_z^2-\frac{\mu}{z^2}\right)\phi,\Pi_{l',m'}\phi'>_{L^2(\mathbb{R}^{3+N})}$$

$$= -<\left(\Delta_{\mathbf{x}}+\partial_z^2-\frac{\mu}{z^2}\right)\phi,\phi'>_{L^2(\mathbb{R}^3\times]0,\infty[)}\delta_{l,l'}\delta_{m,m'}$$

$$= \left\langle\phi,\phi'\right\rangle_{BL_0^1(\mathbb{R}^3\times]0,\infty[)}\delta_{l,l'}\delta_{m,m'},$$

where $\delta_{a,b}$ is the symbol of Kronecker. We conclude that $\Pi_{l,m}$ is a continuous linear map from $BL_0^1(\mathbb{R}^3\times]0,\infty[)$ to $BL^1(\mathbb{R}^{3+N})$, and also that the map (III.2) is continuous from $C^0(\mathbb{R}_t;BL_0^1(\mathbb{R}^3\times]0,\infty[))$ to $C^0(\mathbb{R}_t;BL^1(\mathbb{R}^{3+N}))$, and from $C^0(\mathbb{R}_t;L^2(\mathbb{R}^3\times]0,\infty[))$ to $C^0(\mathbb{R}_t;L^2(\mathbb{R}^{3+N}))$. Hence it is sufficient to establish that $\Psi := \Pi_{l,m}\Phi$ satisfies (III.4) and (III.5) when $\Phi_0 = \Phi(0,.)$ and $\Phi_1 = \partial_t\Phi(0,.)$ belong to $C_0^\infty\left(\mathbb{R}_{\mathbf{x}}^3\times]0,\infty[_z\right)$. For such a solution, we consider the solution $\hat{\Phi}$ of the Cauchy problem for the wave equation in the $1+3+N$ dimensional Minkowski space :

$$\partial_t^2\hat{\Phi} - \Delta_{\mathbf{x},\mathbf{z}}\hat{\Phi} = 0 \;\; in \;\; \mathbb{R}_{t,\mathbf{x},\mathbf{z}}^{1+3+N},\;\; \hat{\Phi}(0) = \Pi_{l,m}\Phi_0,\;\; \partial_t\hat{\Phi}(0) = \Pi_{l,m}\Phi_1.$$

We know that $\hat{\Phi}\in C^\infty(\mathbb{R}_t;C_0^\infty(\mathbb{R}_{\mathbf{x},\mathbf{z}}^{3+N}))$ and the proof will be achieved if we prove that Ψ is equal to $\hat{\Phi}$. We denote $\Pi_{l,m}^*$ the adjoint of $\Pi_{l,m}$, defined for $\psi\in L^2_{loc}\left(\mathbb{R}_{\mathbf{x}}^3\times\mathbb{R}_{\mathbf{z}}^N\right)$ by

$$\Pi_{l,m}^*\psi(\mathbf{x},z) := z^{\frac{N-1}{2}}\int_{S^{N-1}}\psi(\mathbf{x},z\boldsymbol{\omega})\overline{Y_{l,m}(\boldsymbol{\omega})}d\boldsymbol{\omega}.$$

For all $\phi\in L^2(\mathbb{R}_{\mathbf{x}}^3\times]0,\infty[_z)$ and all $\psi\in L^2_{loc}\left(\mathbb{R}_{\mathbf{x}}^3\times\mathbb{R}_{\mathbf{z}}^N\right)$, we have

$$\Pi_{l,m}^*\Pi_{l,m}\phi = \phi,\;\; \psi = \sum_{l,m}\Pi_{l,m}\Pi_{l,m}^*\psi,$$

moreover we can see from (III.7) that for $\psi\in C_0^\infty\left(\mathbb{R}_{\mathbf{x}}^3\times\left(\mathbb{R}_{\mathbf{z}}^N\setminus\{0\}\right)\right)$, we have

$$\parallel\psi\parallel_{BL^1(\mathbb{R}_{\mathbf{x}}^3\times\mathbb{R}_{\mathbf{z}}^N)}^2 = \sum_{l,m}\parallel\Pi_{l,m}^*\psi\parallel_{BL_0^1(\mathbb{R}_{\mathbf{x}}^3\times]0,\infty[_z)}^2.$$

Furthermore if $\psi\in C_0^\infty\left(\mathbb{R}_{\mathbf{x}}^3\times\left(\mathbb{R}_{\mathbf{z}}^N\setminus\{0\}\right)\right)$ and $\theta\in C_0^\infty(\mathbb{R})$, $\theta(z)=1$ for all $\mid z\mid\le 1$, we can easily check that if $N\ge 3$, $(1-\theta(n\mid \mathbf{z}\mid)\psi(\mathbf{x},\mathbf{z}))$ tends to ψ in $BL^1\left(\mathbb{R}_{\mathbf{x}}^3\times\mathbb{R}_{\mathbf{z}}^N\right)$ as $n\to\infty$. We conclude that $C_0^\infty\left(\mathbb{R}_{\mathbf{x}}^3\times\left(\mathbb{R}_{\mathbf{z}}^N\setminus\{0\}\right)\right)$ is dense in $BL^1\left(\mathbb{R}_{\mathbf{x}}^3\times\mathbb{R}_{\mathbf{z}}^N\right)$ and $\Pi_{l,m}^*$ can be extended into a bounded linear map from $BL^1\left(\mathbb{R}_{\mathbf{x}}^3\times\mathbb{R}_{\mathbf{z}}^N\right)$ to $BL_0^1(\mathbb{R}_{\mathbf{x}}^3\times]0,\infty[_z)$. In the case $N=2$ and $l\ge 1$, $m=\pm 1$, we remark that

$$\partial_z\Pi_{l,m}^*\psi(\mathrm{x},z) = \frac{1}{l}\sqrt{z}\int_0^{2\pi}\left(\sin\theta\partial_{Z_1}\psi(\mathrm{x},z\cos\theta,z\sin\theta) - (\cos\theta\partial_{Z_2}\psi(\mathrm{x},z\cos\theta,z\sin\theta)\right)e^{iml\theta}d\theta,$$

hence $\Pi_{l,m}^*\psi \in BL^1(\mathbb{R}_{\mathbf{x}}^3\times]0,\infty[_z)$ and since $\Pi_{l,m}^*\psi(\mathbf{x},0) = 0$, $\Pi_{l,m}^*$ can be extended into a bounded linear map from $BL^1(\mathbb{R}_{\mathbf{x}}^3 \times \mathbb{R}_{\mathbf{z}}^2)$ to $BL_0^1(\mathbb{R}_{\mathbf{x}}^3\times]0,\infty[_z)$ again. Therefore we can put $F(t,\mathbf{x},z) := \Pi_{l,m}^*\hat{\Phi}(t,\mathbf{x},z)$. We remark that $F \in C^0(\mathbb{R}_t; BL_0^1(\mathbb{R}_{\mathbf{x}}^3\times]0,\infty[_z))\cap C^1(\mathbb{R}_t; L^2(\mathbb{R}_{\mathbf{x}}^3\times]0,\infty[_z))$, $F(0,\mathbf{x},z) = \Phi_0(\mathbf{x},z)$, $\partial_t F(0,\mathbf{x},z) = \Phi_1(\mathbf{x},z)$. Thus F satisfies (II.9) and (II.13). Now we calculate

$$\left(\partial_t^2 - \Delta_{\mathbf{x}} - \partial_z^2 + \frac{\mu}{z^2}\right) F(t,\mathbf{x},z) = z^{\frac{N-1}{2}} \int_{S^{N-1}} (\partial_t^2\hat{\Phi} - \Delta_{\mathbf{x},\mathbf{z}}\hat{\Phi})(t,\mathbf{x},z\boldsymbol{\omega})\overline{Y_{l,m}(\boldsymbol{\omega})}d\boldsymbol{\omega} = 0.$$

We conclude that $F = \Phi$, and so

$$\Psi = \Pi_{l,m}F.$$

To end the proof we introduce the projector

$$P_{l,m} : \ \psi \in L_{loc}^2(\mathbb{R}_{\mathbf{x}}^3 \times \mathbb{R}_{\mathbf{z}}^N) \longmapsto P_{l,m}\psi(\mathbf{x},\mathbf{z}) := \left(\int_{S^{N-1}} \psi(\mathbf{x}, \mid \mathbf{z} \mid \boldsymbol{\omega})\overline{Y_{l,m}(\boldsymbol{\omega})}d\boldsymbol{\omega}\right) Y_{l,m}\left(\frac{\mathbf{z}}{\mid \mathbf{z} \mid}\right).$$

It satisfies $\Delta_{\mathbf{x},\mathbf{z}}P_{l,m} = P_{l,m}\Delta_{\mathbf{x},\mathbf{z}}$ on $C_0^\infty(\mathbb{R}^{3+N})$. Thus $P_{l,m}\hat{\Phi} \in C^\infty(\mathbb{R}_t; C_0^\infty(\mathbb{R}_{\mathbf{x},\mathbf{z}}^{3+N}))$ is solution of

$$\partial_t^2 P_{l,m}\hat{\Phi} - \Delta_{\mathbf{x},\mathbf{z}}P_{l,m}\hat{\Phi} = 0 \ \ in \ \ \mathbb{R}_{t,\mathbf{x},\mathbf{z}}^{1+3+N}, \ \ P_{l,m}\hat{\Phi}(0) = \Pi_{l,m}\Phi_0, \ \ \partial_t P_{l,m}\hat{\Phi}(0) = \Pi_{l,m}\Phi_1.$$

By uniqueness, we deduce that $P_{l,m}\hat{\Phi} = \hat{\Phi}$. Since $\Pi_{l,m}F = P_{l,m}\hat{\Phi}$, we finally get $\Psi = \hat{\Phi}$ and that achieves the proof.

For the sake of completeness, we mention another way to obtain this result. We could use the expansion (II.15) and show that $(\partial_t^2 - \Delta_{\mathbf{x},z})\left[\phi_m(t,\mathbf{x}) \mid z \mid^{-\frac{N}{2}+1} J_\lambda(m \mid z \mid)\right] = 0$ when $\lambda = \frac{N}{2} - 1$. The crucial fact is that $-\Delta_z\left[\mid z \mid^{-\frac{N}{2}+1} J_\lambda(m \mid z \mid)\right] = m^2 \mid z \mid^{-\frac{N}{2}+1} J_\lambda(m \mid z \mid)$ in $\mathbb{R}_z^N$.

$$Q.E.D.$$

With this result, we can easily deduce the properties of the finite energy solutions when μ can be expressed by (III.1), from the properties of the free waves in the Minkowski space-time. We use just the formula

(III.8) $$\Phi(t,\mathbf{x},z) = z^{\frac{N-1}{2}} \int_{S^{N-1}} \Psi(t,\mathbf{x},z\boldsymbol{\omega})\overline{Y_{l,m}(\boldsymbol{\omega})}d\boldsymbol{\omega},$$

where Ψ is the solution of (III.4), (III.5) with

(III.9) $$\Psi(0) = \Pi_{l,m}\Phi_0, \ \ \partial_t\Psi(0) = \Pi_{l,m}\Phi_1.$$

The first statement deals with the existence of a lacuna, and also the equipartition of the energy at finite time when the initial data is compactly supported and ν is even.

Theorem III.2. *We assume that*

(III.10) $$\mu = n^2 - \frac{1}{4}, \ \ n \in \mathbb{N}^*,$$

and $\Phi_0 \in BL_0^1(\mathbb{R}^3\times]0,\infty[)$, $\Phi_1 \in L^2(\mathbb{R}^3\times]0,\infty[)$ satisfy for some $R > 0$

(III.11) $$\mid \mathbf{x} \mid^2 +z^2 \leq R^2 \Rightarrow \Phi_0(\mathbf{x},z) = \Phi_1(\mathbf{x},z) = 0.$$

Then the finite energy solution Φ of (I.3), (II.9) and (II.13), satisfies

(III.12) $$\mid \mathbf{x} \mid^2 +z^2 \leq (\mid t \mid -R)^2 \Rightarrow \Phi(t,\mathbf{x},z) = 0,$$

and for $\mid t \mid \geq R$ the potential and kinetic energies are equal :

(III.13) $$\int_{\mathbb{R}^3} \int_0^\infty \mid \nabla_{\mathbf{x},z}\Phi(t,\mathbf{x},z) \mid^2 +\frac{\mu}{z^2} \mid \Phi(t,\mathbf{x},z) \mid^2 d\mathbf{x}dz = \int_{\mathbb{R}^3} \int_0^\infty \mid \partial_t\Phi(t,\mathbf{x},z) \mid^2 d\mathbf{x}dz.$$

These properties are somewhat unexpected : the Huygens Principle fails for the equation (I.3) since the number of the space dimensions is even and the Hadamard's criterion is not satisfied (see [12], Theorem 1.3, p.231; we also note that (I.3) has the form of the equations considered by K.L Stellmacher that are Huygens operators iff the space dimension is odd, [12] chapter V.4). Therefore we must not confuse the existence of this lacuna with the Huygens Principle that is a much stronger property that is not satisfied in our case. A similar situation occurs for the very simple case of the wave equation on the half line, $\partial_t^2 u - \partial_z^2 u = 0$, $z > 0$, with the Dirichlet condition $u(t, z = 0) = 0$. In particular, the previous result assures that there exists a lacuna for the finite energy gravitational fluctuations, since $\mu = \frac{15}{4} = 2^2 - \frac{1}{4}$, and the electromagnetic fields since $\mu = \frac{3}{4} = 1^2 - \frac{1}{4}$. At our knowledge, this property of the gravitational or electromagnetic waves in AdS^5 was not mentioned in the literature. To a similar property of "characteristic propagation" in AdS^4, see [25]. The equipartition of the energy at finite time is also rather surprising since it is a well-known result for the free waves just for the odd space dimension [6].

Proof of Theorem III.2. These results are direct consequences of (III.8). We take $N = 2(n+1)$ and $l = 0$, and to get (III.12), we invoke the well known Huygens Principle satisfied by the free waves $\Psi(t, .) := (\Pi_{0,0}\Phi(t, .))$ in the Minkowski space-time $\mathbb{R}_t \times \mathbb{R}_{\mathbf{x}, \mathbf{z}}^{3+2(n+1)}$. To obtain (II.2), we use (II.11) with $\alpha = n + \frac{1}{2}$ and we check that

$$\int_{\mathbb{R}^3} \int_0^\infty \mid \nabla_{\mathbf{x}, z} \Phi(t, \mathbf{x}, z) \mid^2 + \frac{\mu}{z^2} \mid \Phi(t, \mathbf{x}, z) \mid^2 - \mid \partial_t \Phi(t, \mathbf{x}, z) \mid^2 d\mathbf{x} dz$$
$$= \frac{1}{\mid S^{2n+1} \mid} \int_{\mathbb{R}^{3+2n+2}} \mid \nabla_{\mathbf{x}, \mathbf{z}} \Psi(t, \mathbf{x}, \mathbf{z}) \mid^2 - \mid \partial_t \Psi(t, \mathbf{x}, \mathbf{z}) \mid^2 d\mathbf{x} d\mathbf{z}.$$

It is a classical result (Lax-Phillips, Duffin, see e.g. [6]) that the last integral is zero when $\mid t \mid \geq R$, hence (III.13) follows.

Q.E.D.

Now we prove that the weak Huygens principle holds, that is to say, the singularities are propagating according to the geometrical optics, in particular, they are reflected by the horizon $z = 0$ with the Descartes law. The following proposition describes the structure of the wave front set $WF(\Phi)$ of a finite energy solution. Since the principal part of the differential operator (I.3) is simply the wave equation in the flat space, we know that

$$WF(\Phi) \subset \left\{ (t, \mathbf{x}, z; \tau, \boldsymbol{\xi}, \zeta) \in \mathbb{R} \times \mathbb{R}^3 \times]0, \infty[\times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}; \ \tau^2 = \mid \boldsymbol{\xi} \mid^2 + \zeta^2 \right\},$$

and we are mainly concerned by the rays $(t + \lambda\tau, \mathbf{x} - \lambda\boldsymbol{\xi}, z - \lambda\zeta)_{\lambda \in \mathbb{R}} \subset \mathbb{R} \times \mathbb{R}^3 \times]0, \infty[$.

Theorem III.3. *We assume that μ satisfies*

$$\mu = \frac{\nu^2 - 1}{4}, \ \ \nu \in \mathbb{N}^*.$$

We consider a finite energy solution Φ, and

$$(t, \mathbf{x}, z; \tau, \boldsymbol{\xi}, \zeta) \in WF(\Phi).$$

Then for any $\lambda \in \mathbb{R}$ such that $\lambda\zeta \neq z$, we have

$$\left(t + \lambda\tau, \mathbf{x} - \lambda\boldsymbol{\xi}, \mid z - \lambda\zeta \mid; \tau, \boldsymbol{\xi}, \frac{z - \lambda\zeta}{\mid z - \lambda\zeta \mid}\zeta \right) \in WF(\Phi).$$

In particular, this thorem explains the role of the horizon in the propagation of the finite energy gravitational fluctuations ($\nu = 4$), and the electromagnetic waves ($\nu = 2$) : the boundary of the Anti-De Sitter universe is a perfect mirror. Nevertheless, the constraint on the mass is somewhat

unsatisfactory, and we could expect that the time-like horizon acts like a perfectly reflecting bound-
ary for all the fields regardless of their mass. Moreover, we can hope that the time-like horizons of
the general space-times that are asympotically Anti-de Sitter, have this same property. In a recent
work, A. Vasy has proved this result for the D'Alembertian [26].

Proof of Theorem III.3. We fix $N = \nu + 2$ and we define on $\mathbb{R}_t \times \mathbb{R}^3_{\mathbf{x}} \times \left(\mathbb{R}^N_{\mathbf{z}} \setminus \{0\}\right)$, $\Psi_0(t, \mathbf{x}, \mathbf{z}) :=$
$\Phi(t, \mathbf{x}, |\mathbf{z}|)$. We introduce the map $f(t, \mathbf{x}, \mathbf{z}) := (t, \mathbf{x}, |\mathbf{z}|)$, hence $\Psi_0 = \Phi \circ f := f^*\Phi$ and the
theorem on the wave front set of a pullback (see e.g. Theorem 8.2.4 in [14]) assures that

$$WF(\Psi_0) \subset f^*WF(\Phi) := \left\{ \left(t, \mathbf{x}, \mathbf{z}; {}^t f'(t, \mathbf{x}, \mathbf{z})(\tau, \boldsymbol{\xi}, \zeta)\right); \ (t, \mathbf{x}, |\mathbf{z}|; \tau, \boldsymbol{\xi}, \zeta)) \in WF(\Phi) \right\}$$

therefore

$$(\text{III.14}) \qquad WF(\Psi_0) \subset \left\{ \left(t, \mathbf{x}, \mathbf{z}; \tau, \boldsymbol{\xi}, \zeta \frac{\mathbf{z}}{|\mathbf{z}|}\right); \ (t, \mathbf{x}, |\mathbf{z}|; \tau, \boldsymbol{\xi}, \zeta)) \in WF(\Phi) \right\}.$$

We consider also the function $g : \mathbb{R}_t \times \mathbb{R}^3_{\mathbf{x}} \times], \infty[_z \to \mathbb{R}_t \times \mathbb{R}^3_{\mathbf{x}} \times \left(\mathbb{R}^N_{\mathbf{z}} \setminus \{0\}\right)$ given by $g(t, \mathbf{x}, z) =$
$(t, \mathbf{x}, z, 0_{\mathbb{R}^{N-1}})$. We have $\Phi = g^*\Psi_0$ and by the same theorem

$$WF(\Phi) \subset g^*(WF(\Psi_0)) = \{(t, \mathbf{x}, z; \boldsymbol{\xi}, \zeta_1); \ (t, \mathbf{x}, z, 0_{\mathbb{R}^{N-1}}; \tau, \boldsymbol{\xi}, \boldsymbol{\zeta}) \in WF(\Psi_0)\},$$

hence by using (III.14) we see that $\boldsymbol{\zeta} = (\zeta, 0_{\mathbb{R}^{N-1}})$, and so

$$(\text{III.15}) \qquad WF(\Phi) \subset \{(t, \mathbf{x}, z; \tau, \boldsymbol{\xi}, \zeta); \ (t, \mathbf{x}, z, 0_{\mathbb{R}^{N-1}}; \tau, \boldsymbol{\xi}, \zeta, 0_{\mathbb{R}^{N-1}}) \in WF(\Psi_0)\}.$$

We conclude from (III.14) and (III.15) that

$$(\text{III.16}) \qquad (t, \mathbf{x}, |z|; \tau, \boldsymbol{\xi}, \zeta) \in WF(\Phi) \iff \left(t, \mathbf{x}, z, 0_{\mathbb{R}^{N-1}}; \tau, \boldsymbol{\xi}, \zeta \frac{z}{|z|}, 0_{\mathbb{R}^{N-1}}\right) \in WF(\Psi_0)$$

We now consider $\Psi(t, \mathbf{x}, \mathbf{z}) := |\mathbf{z}|^{-\frac{N-1}{2}} \Psi_0(t, \mathbf{x}, \mathbf{z})$. We have $WF(\Psi) = WF(\Psi_0)$. Moreover the
Lemma III.1 says that Ψ is solution of the free wave equation in the whole Minkowski space-time
$\mathbb{R}_t \times \mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}^N_{\mathbf{z}}$. Then the theorem of the propagation of the singularities (Theorem 8.3.3 in[14])
assures that $\left(t, \mathbf{x}, z, 0_{\mathbb{R}^{N-1}}; \tau, \boldsymbol{\xi}, \zeta \frac{z}{|z|}, 0_{\mathbb{R}^{N-1}}\right) \in WF(\Psi)$ iff $\forall \lambda \in \mathbb{R}$,

$$\left(t + \lambda\tau, \mathbf{x} - \lambda\boldsymbol{\xi}, z - \lambda\zeta, 0_{\mathbb{R}^{N-1}}; \tau, \boldsymbol{\xi}, \zeta \frac{z}{|z|}, 0_{\mathbb{R}^{N-1}}\right) \in WF(\Psi).$$

Now the result follows from (III.16).

$$Q.E.D.$$

We end this part with some results of decay. Such properties are important to investigate the
possible stability of the space-time with respect to the gravitational fluctuations. The following
asymptotic behaviours are straightly deduced from the sharp estimates for the free wave equation
established by P. d'Ancona, V. Georgiev and H. Kubo [2]. It will be useful to introduce some
weighted Sobolev spaces in the spirit of Y. Choquet-Bruhat and D. Christodoulou [9] to impose
some constraints at the space infinity and at the horizon $z = 0$. Given an integer $s \geq 0$ and a real
δ, we define the space $H^{s,\delta}_0(\mathbb{R}^3_{\mathbf{x}} \times]0, \infty[_z)$ as the completion of $C^\infty_0 (\mathbb{R}^3_{\mathbf{x}} \times]0, \infty[_z)$ for the norm :

$$(\text{III.17}) \qquad \| \Phi \|^2_{H^{s,\delta}_0} := \sum_{|\alpha| \leq s} \sum_{k=0}^{s-|\alpha|} \sum_{b=0}^{k} \| \left(z^k + z^{b-k}\right)\left(1 + |\mathbf{x}|^2 + z^2\right)^{\frac{\delta+|\alpha|+k}{2}} \partial^\alpha_{\mathbf{x}} \partial^b_z \Phi \|^2_{L^2(\mathbb{R}^3_{\mathbf{x}} \times]0, \infty[_z)}.$$

Theorem III.4. *The finite energy solution of (I.3) and (II.13) with $\mu = \frac{\nu^2-1}{4}$, $\nu \in \mathbb{N}^*$, satisfies the following estimates for all $\epsilon > 0$ and all $\mathrm{x} \in \mathbb{R}^3$, $z > 0$, provided the right members are finite :*

(III.18)
$$\left(1+ \mid t \mid + \mid \mathrm{x} \mid +z+ \mid t^2- \mid \mathrm{x} \mid^2 -z^2 \mid \right)^{\frac{\nu}{2}+2} z^{-\frac{\nu+1}{2}} \mid \Phi(t,\mathrm{x},z) \mid$$

$$\leq C(\epsilon) \left(\parallel \Phi_0 \parallel_{H_0^{[\frac{\nu+7}{2}],\frac{\nu+3}{2}+\epsilon}} + \parallel \Phi_1 \parallel_{H_0^{[\frac{\nu+5}{2}],\frac{\nu+5}{2}+\epsilon}} \right),$$

$$(1+ \mid t \mid + \mid \mathrm{x} \mid +z)^{\frac{\nu}{2}+2} z^{-\frac{\nu+1}{2}} \mid \Phi(t,\mathrm{x},z) \mid$$
(III.19)
$$\leq C(\epsilon) \left(\parallel \Phi_0 \parallel_{H_0^{[\frac{\nu+7}{2}],-\frac{1}{2}+\epsilon}} + \parallel \Phi_1 \parallel_{H_0^{[\frac{\nu+5}{2}],\frac{1}{2}+\epsilon}} \right),$$

(III.20) $\qquad \parallel (1+ \mid t \mid + \mid \mathrm{x} \mid +z)^{-\frac{1}{2}-\epsilon}\Phi \parallel_{L^2(\mathbb{R}_t \times \mathbb{R}_{\mathrm{x}}^3 \times]0,\infty[_z)} \leq C(\epsilon) \left(\parallel \Phi_0 \parallel_{H_0^{1,\epsilon}} + \parallel \Phi_1 \parallel_{H_0^{0,1+\epsilon}} \right),$

where the constant $C(\epsilon)$ is independent of Φ_0 and Φ_1.

These are much stronger than the theorem II.38, since we get a uniform decay in space of $z^{-\frac{1}{2}-\sqrt{\mu+\frac{1}{4}}}\Phi(t,.)$ as $t^{-2-\sqrt{\mu+\frac{1}{4}}}$ that increases with the mass, instead of $t^{-\frac{3}{2}}$. We can expect that this rate of decay is not due to the peculiar form of the mass, and that remains true for any $\mu > -\frac{1}{4}$. These estimates are not optimal with respect to the norms that appear in the right members. The functional framework could be improved by introducing Sobolev spaces $H_0^{s,\delta}$ with non integer exponent s by interpolation that could allow also to obtain many other inequalities more precise. This work should be useful to investigate the non linear problems arising in the Anti-de Sitter cosmology.

Proof of Theorem III.4. It is sufficient to treat the case $\Phi_0, \Phi_1 \in C_0^\infty (\mathbb{R}_{\mathrm{x}}^3 \times]0,\infty[_z)$. We consider Ψ defined by (III.2) with $N = \nu + 2$ and $l = 0$ and we put $\Psi_j(\mathrm{x},z) := \mid z \mid^{-\frac{\nu+1}{2}} \Phi_j(\mathrm{x}, \mid z \mid)$. We use Theorem 1.1 of [2] with $n = \nu + 5$ to get with $d = \frac{n-1}{2}$,

$$\left(1+ \mid t \mid + \mid \mathrm{x} \mid + \mid z \mid + \mid t^2- \mid \mathrm{x} \mid^2 - \mid z \mid^2 \mid \right)^{\frac{\nu}{2}+2} \mid \Psi(t,\mathrm{x},z) \mid$$

$$\leq C(\epsilon) \left(\parallel \Psi_0 \parallel_{H^{[\frac{\nu+7}{2}],\frac{\nu+3}{2}+\epsilon}} + \parallel \Psi_1 \parallel_{H^{[\frac{\nu+5}{2}],\frac{\nu+5}{2}+\epsilon}} \right),$$

and with $n = 0$,

$$\mid \Psi(t,\mathrm{x},z) \mid \leq C(\epsilon)(1+ \mid t \mid + \mid \mathrm{x} \mid + \mid z \mid)^{-\frac{\nu}{2}-2} \left(\parallel \Psi_0 \parallel_{H^{[\frac{\nu+7}{2}],-\frac{1}{2}+\epsilon}} + \parallel \Psi_1 \parallel_{H^{[\frac{\nu+5}{2}],\frac{1}{2}+\epsilon}} \right),$$

and the Theorem 1.3 of this paper with $q = 2$, $\rho = -\frac{1}{2} - \epsilon$, $\sigma = 0$, $n = \nu + 5$ assures that

$$\parallel (1+ \mid t \mid + \mid \mathrm{x} \mid + \mid z \mid)^{-\frac{1}{2}-\epsilon}\Psi \parallel_{L^2(\mathbb{R}_t \times \mathbb{R}_{\mathrm{x}}^3 \times \mathbb{R}_z^N)} \leq C(\epsilon) (\parallel \Psi_0 \parallel_{H^{1,\epsilon}} + \parallel \Psi_1 \parallel_{H^{0,1+\epsilon}}).$$

Here the constant $C(\epsilon) > 0$ is independent of Ψ_0 and Ψ_1 and the $H^{s,\delta}$ norms on $\mathbb{R}^{3+N}$ are defined by :

(III.21) $\qquad \parallel \psi \parallel_{H^{s,\delta}}^2 := \sum_{|\alpha|+|\beta|\leq s} \parallel \left(1+ \mid \mathrm{x} \mid^2 + \mid z \mid^2 \right)^{\frac{\delta+|\alpha|+|\beta|}{2}} \partial_{\mathrm{x}}^\alpha \partial_z^\beta \psi \parallel_{L^2(\mathbb{R}_{\mathrm{x}}^3 \times \mathbb{R}_z^N)}^2 .$

We can check by iteration on $\beta \in \mathbb{N}^N$ that for all $\phi \in C_0^\infty (\mathbb{R}_{\mathrm{x}}^3 \times]0,\infty[_z)$, we have $\partial_{\mathrm{x}}^\alpha \partial_z^\beta \left[\mid z \mid^{-\frac{N-1}{2}} \phi(\mathrm{x}, \mid z \mid) \right] = \sum_{finite} P_c(z) \mid z \mid^{-\frac{N-1}{2}-a} \partial_{\mathrm{x}}^\alpha \partial_z^b \phi(\mathrm{x}, \mid z \mid)$, with $a,b,c \in \mathbb{N}$, $P_c \in \mathbb{R}[z]$, $d^\circ P_c \leq c$, $b \leq \mid \beta \mid$, $c \leq \mid \beta \mid$, $a \leq \mid \beta \mid -b+c$. We deduce that there exists $K > 0$ such that for all $(\mathrm{x},z) \in \mathbb{R}^{3+N}$ we have

$$\left| \partial_{\mathrm{x}}^\alpha \partial_z^\beta \left[\mid z \mid^{-\frac{N-1}{2}} \phi(\mathrm{x}, \mid z \mid) \right] \right| \leq K \mid z \mid^{-\frac{N-1}{2}} \sum_{b\leq|\beta|} \left(\mid z \mid^{|\beta|} + \mid z \mid^{b-|\beta|} \right) \left| \partial_{\mathrm{x}}^\alpha \partial_z^b \phi(\mathrm{x}, \mid z \mid) \right|.$$

We conclude that

(III.22)
$$\| \, | \, z \, |^{-\frac{N-1}{2}} \, \phi(x, | \, z \, |) \, \|_{H^{s,\delta}} \lesssim \| \, \phi \, \|_{H_0^{s,\delta}}$$

and the theorem follows from the previous inequalities on Ψ.

Q.E.D.

We end these results on the asymptotic behaviours with some global estimates of Strichartz type. We present just the cases for which the energy allows to control the L^p-norms.

Theorem III.5. *The finite energy solution of (I.3) and (II.13) with $\mu = \frac{\nu^2 - 1}{4}$, $\nu \in \mathbb{N}^*$, satisfies the following estimate*

(III.23)
$$\| \, z^{(\nu+1)\left(\frac{1}{r} - \frac{1}{2}\right)} \Phi \, \|_{L^q(\mathbb{R}_t; L^r(\mathbb{R}_x^3 \times]0, \infty[_z))} \lesssim \left(\| \, \Phi_0 \, \|_{BL_0^1(\mathbb{R}_x^3 \times]0, \infty[_z)} + \| \, \Phi_1 \, \|_{L^2(\mathbb{R}_x^3 \times]0, \infty[_z)} \right)$$

when

(III.24)
$$2 \leq q, \quad \frac{1}{q} + \frac{\nu + 5}{r} = \frac{\nu + 3}{2}, \quad \frac{1}{q} + \frac{\nu + 4}{2r} \leq \frac{\nu + 4}{4}.$$

Moreover, if $\nu = 2k$, $k \in \mathbb{N}^*$, then

(III.25)
$$\| \, z^{\left(\frac{1}{r} - \frac{1}{2}\right)} \Phi \, \|_{L^q(\mathbb{R}_t; L^r(\mathbb{R}_x^3 \times]0, \infty[_z))} \lesssim \left(\| \, \Phi_0 \, \|_{BL_0^1(\mathbb{R}_x^3 \times]0, \infty[_z)} + \| \, \Phi_1 \, \|_{L^2(\mathbb{R}_x^3 \times]0, \infty[_z)} \right)$$

when

(III.26)
$$2 \leq q, \quad \frac{1}{q} + \frac{5}{r} = \frac{3}{2}, \quad \frac{1}{q} + \frac{2}{r} \leq 1,$$

and if $\nu = 2k + 1$, $k \in \mathbb{N}$, then

(III.27)
$$\| \, z^{\left(\frac{2}{r} - 1\right)} \Phi \, \|_{L^q(\mathbb{R}_t; L^r(\mathbb{R}_x^3 \times]0, \infty[_z))} \lesssim \left(\| \, \Phi_0 \, \|_{BL_0^1(\mathbb{R}_x^3 \times]0, \infty[_z)} + \| \, \Phi_1 \, \|_{L^2(\mathbb{R}_x^3 \times]0, \infty[_z)} \right)$$

when

(III.28)
$$2 \leq q, \quad \frac{1}{q} + \frac{6}{r} = 2, \quad \frac{1}{q} + \frac{5}{2r} \leq \frac{5}{4}.$$

In the case of the gravitational fluctuations, we have $\nu = 4$ and we can control the weighted global L^q norms by the energy :

$$\| \, z^{-\frac{3}{4}} \Phi \, \|_{L^{\frac{20}{7}}(\mathbb{R}_t \times \mathbb{R}_x^3 \times]0, \infty[_z)} + \| \, z^{-\frac{1}{4}} \Phi \, \|_{L^4(\mathbb{R}_t \times \mathbb{R}_x^3 \times]0, \infty[_z)} \lesssim \left(\| \, \Phi_0 \, \|_{BL_0^1(\mathbb{R}_x^3 \times]0, \infty[_z)} + \| \, \Phi_1 \, \|_{L^2(\mathbb{R}_x^3 \times]0, \infty[_z)} \right).$$

For the electromagnetic fluctuations we have $\nu = 2$ and

$$\| \, z^{-\frac{9}{16}} \Phi \, \|_{L^{\frac{16}{5}}(\mathbb{R}_t \times \mathbb{R}_x^3 \times]0, \infty[_z)} + \| \, z^{-\frac{1}{4}} \Phi \, \|_{L^4(\mathbb{R}_t \times \mathbb{R}_x^3 \times]0, \infty[_z)} \lesssim \left(\| \, \Phi_0 \, \|_{BL_0^1(\mathbb{R}_x^3 \times]0, \infty[_z)} + \| \, \Phi_1 \, \|_{L^2(\mathbb{R}_x^3 \times]0, \infty[_z)} \right).$$

Proof of Theorem III.5. We recall that the famous Strichartz estimates sharpened in [17], state that the finite energy solutions Ψ of the wave equation on the Minkowski space-time $\mathbb{R}^{1+n}$ belongs to $L^q(\mathbb{R}_t; L^r(\mathbb{R}^n))$ iff

(III.29)
$$\frac{1}{q} + \frac{n}{r} = \frac{n-2}{2}, \quad \frac{1}{q} + \frac{n-1}{2r} \leq \frac{n-1}{4},$$

and this norm is controled by the energy :

$$\| \, \Psi \, \|_{L^q(\mathbb{R}_t; L^r(\mathbb{R}^n))} \lesssim \left(\| \, \Psi(0, .) \, \|_{BL^1(\mathbb{R}^n)} + \| \, \partial_t \Psi(0, .) \, \|_{L^2(\mathbb{R}^n)} \right).$$

When Ψ is given by (III.2), where N and μ are related by (III.1), we have $n = N + 3$ and

$$\| \, \Psi \, \|_{L^q(\mathbb{R}_t; L^r(\mathbb{R}^{N+3}))} = \| \, Y_{l,m} \, \|_{L^r(S^{N-1})} \| \, z^{(N-1)\left(\frac{1}{r} - \frac{1}{2}\right)} \Phi \, \|_{L^q(\mathbb{R}_t; L^r(\mathbb{R}_x^3 \times]0, \infty[_z))},$$

$$\| \, \Psi(0, .) \, \|_{BL^1(\mathbb{R}^{N+3})} = \| \, \Phi_0 \, \|_{BL_0^1(\mathbb{R}_x^3 \times]0, \infty[_z)}, \quad \| \, \partial_t \Psi(0, .) \, \|_{L^2(\mathbb{R}^{N+3})} = \| \, \Phi_1 \, \|_{L^2(\mathbb{R}_x^3 \times]0, \infty[_z)}.$$

We deduce that

(III.30) $\qquad \| z^{(N-1)\left(\frac{1}{r}-\frac{1}{2}\right)}\Phi \|_{L^q(\mathbb{R}_t; L^r(\mathbb{R}^3_{\mathbf{x}}\times]0,\infty[_z))} \lesssim \left(\| \Phi_0 \|_{BL^1_0(\mathbb{R}^3_{\mathbf{x}}\times]0,\infty[_z)} + \| \Phi_1 \|_{L^2(\mathbb{R}^3_{\mathbf{x}}\times]0,\infty[_z)} \right).$

(III.1) allows to choose N and l such that

$$\nu = N + 2l - 2, \ \ N \geq 3 \ and \ l \geq 0, \ or \ N = 2 \ and \ l \geq 1.$$

First we choose $l = 0$, and $N = \nu + 2$. Then (III.23) and (III.24) follow from (III.30) and (III.29) with $n = \nu + 5$. Now when $\nu = 2k$, $k \in \mathbb{N}^*$, we take $l = k$, $N = 2$ hence (III.25) and (III.26) follow from (III.30) and (III.29) with $n = 5$. Finally when $\nu = 2k + 1$, $k \in \mathbb{N}$, we take $l = k$, $N = 3$ hence (III.27) and (III.28) follow from (III.30) and (III.29) with $n = 6$.

$$Q.E.D.$$

IV. NORMALIZABLE SOLUTIONS IN BRANE COSMOLOGY

In brane cosmology, the Minkowski space-time $\mathbb{R}_t \times \mathbb{R}^3_{\mathbf{x}}$ is considered as a brane that is the boundary $\mathbb{R}_t \times \mathbb{R}^3_{\mathbf{x}} \times \{z = 1\}$ of a part $\mathcal{B}$ of AdS^5 called the bulk. The choice of the bulk depends on the tension of this brane (see [19]). The $RS2$ Randall-Sundrum model that we have investigated in [5], deals with the Minkowski brane with a positive tension associated to the bulk $\mathcal{B} = \mathbb{R}_t \times \mathbb{R}^3_{\mathbf{x}} \times]1, \infty[_z$. In this part we consider the case of the Minkowski brane with a negative tension. In this case $\mathcal{B} = \mathbb{R}_t \times \mathbb{R}^3_{\mathbf{x}} \times]0, 1[_z$ and we have to study the Klein-Gordon equation

(IV.1) $$\left(\partial_t^2 - \Delta_{\mathbf{x}} - \partial_z^2 + \frac{\mu}{z^2} \right) \Phi = 0, \ \ (t, \mathbf{x}, z) \in \mathbb{R} \times \mathbb{R}^3 \times]0, 1[.$$

The boundary condition on the brane is associated to the Z_2 symmetry (see [19] and the Appendix) that yields to the Neumann condition on the fields. With the change of unknown, we finally impose the Robin condition :

(IV.2) $$\partial_z \Phi(t, \mathbf{x}, 1) + \frac{3}{2}\Phi(t, \mathbf{x}, 1) = 0, \ \ t \in \mathbb{R}, \ \ \mathbf{x} \in \mathbb{R}^3.$$

Associated to these constraints, there exists a formally conserved energy

(IV.3) $\qquad E_1(\Phi, t) := \int_{\mathbb{R}^3} \int_0^1 \mid \nabla_{t,\mathbf{x},z}\Phi(t, \mathbf{x}, z) \mid^2 + \frac{\mu}{z^2} \mid \Phi(t, \mathbf{x}, z) \mid^2 d\mathbf{x}dz + \frac{3}{2}\int_{\mathbb{R}^3} \Phi(t, \mathbf{x}, 1)d\mathbf{x}.$

To solve the mixed problem in a suitable functional framework, since $H^1(\mathbb{R}^3_{\mathbf{x}} \times]0, 1[_z)$ is a subspace of $C^0\left([0, 1]_z; H^{\frac{1}{2}}(\mathbb{R}^3_{\mathbf{x}})\right)$, we can introduce the space

(IV.4) $$W^1 := \left\{ \phi \in H^1\left(\mathbb{R}^3_{\mathbf{x}} \times]0, 1[_z\right); \ \phi(\mathbf{x}, 0) = 0 \right\},$$

and we put

(IV.5) $\qquad \|\phi\|_{W^1}^2 := \int_{\mathbb{R}^3} \int_0^1 \mid \nabla_{t,\mathbf{x},z}\phi(\mathbf{x}, z) \mid^2 + \frac{\mu}{z^2} \mid \phi(t, \mathbf{x}, z) \mid^2 d\mathbf{x}dz + \frac{3}{2}\int_{\mathbb{R}^3} \phi(\mathbf{x}, 1)d\mathbf{x}.$

Thanks to the Hardy inequality (II.6), and the continuity of the trace on $z = 1$, $\|.\|_{W^1}$ is a norm on W^1 when $\mu > -\frac{1}{4}$, that is equivalent to the usual H^1-norm. Since W^1 is a closed subspace of $H^1(\mathbb{R}^3_{\mathbf{x}} \times]0, 1[_z)$, we can see that W^1 endowed with the norm $\|.\|_{W^1}$ is a Hilbert space. We need to use also the space

(IV.6) $\qquad W^2 := \left\{ \phi \in W^1; \ \Delta_{\mathbf{x},z}\phi - \frac{\mu}{z^2}\phi \in L^2\left(\mathbb{R}^3_{\mathbf{x}} \times]0, 1[_z\right), \ \partial_z\phi(\mathbf{x}, 1) + \frac{3}{2}\phi(\mathbf{x}, 1) = 0 \right\}.$

This definition makes sense since if $\phi \in W^2$, then $\phi \in H^1\left(\mathbb{R}^3_{\mathbf{x}} \times]0, 1[_z\right)$ and $\Delta_{\mathbf{x},z}\phi \in L^2\left(\mathbb{R}^3_{\mathbf{x}} \times]a, 1[_z\right)$ for all $a \in]0, 1[$. As a consequence, ϕ satisfies $\partial_z\phi \in C^0\left([a, 1]_z; H^{-\frac{1}{2}}(\mathbb{R}^3_{\mathbf{x}})\right)$ hence the boundary

condition on the brane $z = 1$ is well defined. Moreover, W^2 is a Hilbert space for the norm

(IV.7)
$$\|\phi\|_{W^2}^2 := \|\phi\|_{W^1}^2 + \|\Delta_{\mathbf{x},z}\phi - \frac{\mu}{z^2}\phi\|_{L^2}^2.$$

When $\phi \in W^1$ but $\Delta_{\mathbf{x},z}\phi \notin L^2$, the trace $\partial_z\phi(\mathbf{x}, 1)$ does not exist. In order to the boundary condition (IV.2) makes sense, we introduce the space of the W^2-valued distributions on $\mathbb{R}_t$, $\mathcal{D}'(\mathbb{R}_t; W^2)$, that is the set of the linear continuous maps from $C_0^\infty(\mathbb{R}_t)$ to W^2. We say that the boundary condition (IV.2) is satisfied by Φ when $\Phi \in \mathcal{D}'(\mathbb{R}_t; W^2)$.

Theorem IV.1. *Given $\Phi_0 \in W^1$, $\Phi_1 \in L^2\left(\mathbb{R}_{\mathbf{x}}^3 \times]0, 1[_z\right)$, there exists a unique solution Φ of (IV.1) satisfying*

(IV.8)
$$\Phi \in \mathcal{D}'(\mathbb{R}_t; W^2) \cap C^0\left(\mathbb{R}_t; W^1\right) \cap C^1\left(\mathbb{R}_t; L^2\left(\mathbb{R}_{\mathbf{x}}^3 \times]0, 1[_z\right)\right),$$

(IV.9)
$$\Phi(0, \mathbf{x}, z) = \Phi_0(\mathbf{x}, z), \quad \partial_z\Phi(0, \mathbf{x}, z) = \Phi_1(\mathbf{x}, z), \quad (\mathbf{x}, z) \in \mathbb{R}^3 \times]0, 1[.$$

Moreover the energy of Φ is conserved :

(IV.10)
$$\forall t \in \mathbb{R}, \quad E_1(\Phi, t) = E_1(\Phi, 0).$$

When $\Phi_0 \in W^2$ and $\Phi_1 \in W^1$, then

(IV.11)
$$\Phi \in C^0\left(\mathbb{R}_t; W^2\right) \cap C^1\left(\mathbb{R}_t; W^1\right) \cap C^2\left(\mathbb{R}_t; L^2\left(\mathbb{R}_{\mathbf{x}}^3 \times]0, 1[_z\right)\right).$$

Proof of Theorem IV.1. Since μ is a real number, it is sufficient to consider just the real valued solutions, hence, in this proof, we suppose that all the functions in L^2 are real valued and all the spaces are real Hilbert spaces. We introduce the operator on $W^1 \times L^2\left(\mathbb{R}_{\mathbf{x}}^3 \times]0, 1[_z\right)$, given by

$$A := \begin{pmatrix} 0 & 1 \\ \Delta_{\mathbf{x}} + \partial_z^2 - \frac{\mu}{z^2} & 0 \end{pmatrix}, \quad Dom(A) = W^2 \times W^1.$$

We prove that A and $-A$ are maximal monotone operators. Given $u \in W^2$, $v \in W^1$, we get with the Green formula

$$\left\langle A\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_{W^1 \times L^2} = \langle \partial_z u(., 1), v(., 1) \rangle_{H^{-\frac{1}{2}}(\mathbb{R}^3), H^{\frac{1}{2}}(\mathbb{R}^3)} + \frac{3}{2}\int_{\mathbb{R}^3} v(\mathbf{x}, 1)u(\mathbf{x}, 1)d\mathbf{x}$$

$$= \frac{3}{2}\int_{\mathbb{R}^3} v(\mathbf{x}, 1)u(\mathbf{x}, 1) - u(\mathbf{x}, 1)v(\mathbf{x}, 1)d\mathbf{x}$$

$$= 0.$$

On the other hand, given $f \in W^1$, $g \in L^2$, $(u_\pm, v_\pm) \in W^2 \times W^1$ is solution of

$$(A \pm Id)\begin{pmatrix} u_\pm \\ v_\pm \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

iff

$$u_\pm \in W^2, \quad v = f \mp u_\pm, \quad \Delta_{\mathbf{x},z}u_\pm - \left(\frac{\mu}{z^2} + 1\right)u_\pm = g \mp f.$$

This partial differential equation is easily solved by a variational method. We introduce the bilinear form

$$a(u, u') := \int_{\mathbb{R}^3}\int_0^1 \nabla_{\mathbf{x},z}u.\nabla_{\mathbf{x},z}u' + \left(\frac{\mu}{z^2} + 1\right)uu'd\mathbf{x}dz + \frac{3}{2}\int_{\mathbb{R}^3} u(\mathbf{x}, 1)u'(\mathbf{x}, 1)d\mathbf{x}.$$

Since $a(.,.)$ is continuous and coercive on W^1, the Lax-Milgram theorem implies there exists $u_\pm \in W^1$ such that

$$\forall u' \in W^1, \quad a(u_\pm, u') = \int_{\mathbb{R}^3}(g \mp f)u'(\mathbf{x}, 1)d\mathbf{x}.$$

Taking u' any test function, we deduce that $\Delta_{\mathbf{x},z}u_\pm - \left(\frac{\mu}{z^2}+1\right)u_\pm = g \mp f$, and so $\Delta_{\mathbf{x},z}u_\pm - \frac{\mu}{z^2}u_\pm \in L^2$. Then the Green formula gives for all $u' \in W^1$

$$a(u_\pm, u') = \int_{\mathbb{R}^3}\int_0^1 \left(\Delta_{\mathbf{x},z}u_\pm - \left(\frac{\mu}{z^2}+1\right)u_\pm\right)u'd\mathbf{x}dz + \int_{\mathbb{R}^3}\left(\partial_z u_\pm(\mathbf{x},1)+\frac{3}{2}u_\pm(\mathbf{x},1)\right)u'(\mathbf{x},1)d\mathbf{x}.$$

We deduce that $\partial_z u_\pm(\mathbf{x},1)+\frac{3}{2}u_\pm(\mathbf{x},1) = 0$ and $u_\pm \in W^2$. We conclude that $\pm A$ is a densely defined maximal monotone operator on $W^1 \times L^2$, hence the Hille-Yoshida-Phillips theorem assures the existence of a unitary group e^{tA} on $W^1 \times L^2$, so that Φ defined by

$$\begin{pmatrix}\Phi \\ \partial_t\Phi\end{pmatrix} := e^{tA}\begin{pmatrix}\Phi_0 \\ \Phi_1\end{pmatrix}$$

satisfies (IV.1), (IV.11), (IV.9) and (IV.10) when $(\Phi_0,\Phi_1) \in W^2 \times W^1$. When $(\Phi_0,\Phi_1) \in W^1 \times L^2$, we have to show that $\Phi \in \mathcal{D}'(\mathbb{R}_t; W^2)$. We pick a sequence $(\Phi_0^n, \Phi_1^n) \in W^2 \times W^1$ that tends to (Φ_0,Φ_1) in $W^1 \times L^2$ as $n \to \infty$. Then $\begin{pmatrix}\Phi^n \\ \partial_t\Phi^n\end{pmatrix} := e^{tA}\begin{pmatrix}\Phi_0^n \\ \Phi_1^n\end{pmatrix}$ tends to $(\Phi, \partial_t\Phi)$ in $C^0(\mathbb{R}_t; W^1) \cap C^1(\mathbb{R}_t; L^2(\mathbb{R}_{\mathbf{x}}^3 \times]0,1[_z))$. Thus given $\theta \in C_0^\infty(\mathbb{R}_t)$, we have

$$\int \theta(t)\Phi^n(t)dt \to \int \theta(t)\Phi(t)dt \;\; in \;\; W^1.$$

Moreover we have

$$\left(\Delta_{\mathbf{x},z} - \frac{\mu}{z^2}\right)\int \theta(t)\Phi^n(t)dt = \int \theta''(t)\Phi^n(t)dt \to \int \theta''(t)\Phi(t)dt \;\; in \;\; L^2.$$

We conclude that $\int\theta(t)\Phi^n(t)dt$ is a Cauchy sequence in W^2 and $\theta \mapsto \int\theta(t)\Phi(t)dt$ is a W^2-valued distribution on $\mathbb{R}_t$.

To achieve the proof of the theorem and to establish the uniqueness, we show that any Φ solution of (IV.1) and (IV.8), satisfies also (IV.10). We take $\theta \in C_0^\infty(\mathbb{R})$, $0 \le \theta$, $\int\theta(t)dt = 1$, and we put $\Phi^n(t) := n\int\theta(ns)\Phi(t+s)ds$. Φ^n is a solution of (IV.1) and belongs to $C^\infty(\mathbb{R}_t; W^2)$. Then we can multiply the PDE by $\partial_t\Phi^n$ and integrate on $\mathbb{R}_{\mathbf{x}}^3 \times]0,1[_z$ to obtain that $E_1(\Phi^n, t) = E_1(\Phi^n, 0)$. Since Φ^n tends to Φ in $C^0(\mathbb{R}_t; W^1) \cap C^1(\mathbb{R}_t; L^2(\mathbb{R}_{\mathbf{x}}^3 \times]0,1[_z))$ as $n \to \infty$, we conclude that $E_1(\Phi, t) = E_1(\Phi, 0)$.

$$Q.E.D.$$

Following C.H. Wilcox [29], the fields satisfying (IV.8) (respect. (IV.11)), are called *finite energy solutions* (respect. *strict finite energy solutions*). They belong to a larger class of solutions, the so-called *normalizable solutions* that are the fields that are square integrable in space at each time. When the coefficients are smooth up to the boundary, these weak solutions have been studied by Vishik and Ladyzhenskaya [27] and in an abstract setting that allows a time dependence of the coefficients by J-L Lions (chapter 3 of [18]). To define the space of the initial velocity, we introduce the space W^{-1} defined as the dual space of W^1, endowed with its canonical norm. We warn that the elements of this space are not distributions since $C_0^\infty(\mathbb{R}_{\mathbf{x}}^3 \times]0,1[_z)$ is not dense in W^1 ; nevertheless since W^1 is dense in $L^2(\mathbb{R}_{\mathbf{x}}^3 \times]0,1[_z)$, this space can be identified with a subspace of W^{-1}.

Theorem IV.2. *Given $\Phi_0 \in L^2(\mathbb{R}_{\mathbf{x}}^3 \times]0,1[_z)$, $\Phi_1 \in W^{-1}$, there exists a unique solution Φ of (IV.1), (IV.9) satisfying*

(IV.12) $$\Phi \in \mathcal{D}'(\mathbb{R}_t; W^2) \cap C^0\left(\mathbb{R}_t; L^2\left(\mathbb{R}_{\mathbf{x}}^3 \times]0,1[_z\right)\right) \cap C^1\left(\mathbb{R}_t; W^{-1}\right).$$

Furthermore Φ satisfies for all $t \in \mathbb{R}$:

(IV.13) $$\|\partial_t\Phi(t)\|_{W^{-1}}^2 + \|\Phi(t)\|_{L^2}^2 = \|\Phi_1\|_{W^{-1}}^2 + \|\Phi_0\|_{L^2}^2.$$

Proof of Theorem IV.2. We consider the densely defined operator L on $L^2\left(\mathbb{R}^3_{\mathbf{x}}\times\,]0,1[_z\right)$ defined by

(IV.14) $$L := -\Delta_{\mathbf{x}} - \partial_z^2 + \frac{\mu}{z^2}, \quad Dom(L) = W^2.$$

L is symmetric and thanks to the Hardy estimate we have

(IV.15) $$\mu + \frac{1}{4} \leq L.$$

Moreover we have shown in the proof of the previous theorem that $L + 1$ is an isomorphism from W^2 onto L^2. Therefore L is selfadjoint on L^2, $Dom(L^{\frac{1}{2}}) = W^1$ and $L^{-\frac{1}{2}}$ can be uniquely extended in an isometry, again denoted $L^{-\frac{1}{2}}$, from W^{-1} onto L^2. Then Φ given by

(IV.16) $$\Phi(t) := \cos\left(tL^{\frac{1}{2}}\right)\Phi_0 + \sin\left(tL^{\frac{1}{2}}\right)L^{-\frac{1}{2}}\Phi_1$$

belongs to $C^0\left(\mathbb{R}_t; L^2\left(\mathbb{R}^3_{\mathbf{x}}\times\,]0,1[_z\right)\right) \cap C^1\left(\mathbb{R}_t; W^{-1}\right)$ and satisfies (IV.1), (IV.9) and (IV.13). To prove that this function belongs to $\mathcal{D}'(\mathbb{R}_t; W^2)$, we proceed like above. We pick a sequence $(\Phi_0^n, \Phi_1^n) \in W^2 \times W^1$ that tends to (Φ_0, Φ_1) in $L^2 \times W^{-1}$ as $n \to \infty$. Then $\Phi^n(t) := \cos\left(tL^{\frac{1}{2}}\right)\Phi_0^n + \sin\left(tL^{\frac{1}{2}}\right)L^{-\frac{1}{2}}\Phi_1^n$ tends to Φ in $C^0\left(\mathbb{R}_t; L^2\right)$. Thus given $\theta \in C_0^\infty(\mathbb{R}_t)$, $\int \theta(t)\Phi^n(t)dt$ tends to $\int \theta(t)\Phi(t)dt$ in L^2 Moreover we have $L\int \theta(t)\Phi^n(t)dt = \int \theta''(t)\Phi^n(t)dt \to \int \theta''(t)\Phi(t)dt$ *in* L^2. We conclude that $\int \theta(t)\Phi^n(t)dt$ is a Cauchy sequence in W^2 and $\theta \mapsto \int \theta(t)\Phi(t)dt$ is a W^2-valued distribution on $\mathbb{R}_t$. To establish the uniqueness, we show that any Φ solution of (IV.1) and (IV.12), satisfies also (IV.13). We take $\theta \in C_0^\infty(\mathbb{R})$, $0 \leq \theta$, $\int \theta(t)dt = 1$, and we put $\Phi^n(t) := n\int \theta(ns)\Phi(t+s)ds$. Φ^n is a solution of (IV.1) and belongs to $C^\infty(\mathbb{R}_t; W^2)$. Then we can multiply the PDE by $L^{-1}\partial_t\Phi^n$ and integrate on $\mathbb{R}^3_{\mathbf{x}}\times\,]0,1[_z$ to obtain that $\|\partial_t\Phi^n(t)\|_{W^{-1}}^2 + \|\Phi^n(t)\|_{L^2}^2 = \|\Phi_1^n\|_{W^{-1}}^2 + \|\Phi_0^n\|_{L^2}^2$. Since Φ^n tends to Φ in $C^0\left(\mathbb{R}_t; L^2\right) \cap C^1\left(\mathbb{R}_t; W^{-1}\right)$ as $n \to \infty$, we get (IV.13).

$$Q.E.D.$$

Now we express the fields as an expansion of massive Klein-Gordon fields propagating in the Minkowski space-time, the so called *Kaluza-Klein tower*. Unlike the case of the positive-tension brane investigated in [5], for which there exists a continuum of modes, the mass of the modes for the negative-tension brane is quantized and there is no massless gravito. This fact is due to the boundedness of the depth since $z \in\,]0,1[$ instead of $z \in\,]1,\infty[$.

Theorem IV.3. *There exists a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset\,]0,\infty[$ with $\lim_{n\to\infty}\lambda_n = \infty$, and a Hilbert basis of $L^2(0,1)$, $(u_n)_{n\in\mathbb{N}} \subset H^1(]0,1[) \cap C^\infty(]0,1])$ with $u_n(0) = 0$, $u_n'(1) + \frac{3}{2}u_n(1) = 0$, such that for any $\Phi_0 \in L^2(\mathbb{R}^3_{\mathbf{x}}\times\,]0,1[_z)$, $\Phi_1 \in W^{-1}$, the normalizable solution Φ of (IV.1), (IV.9), (IV.12) can be written as*

(IV.17) $$\Phi(t,\mathbf{x},z) = \sum_0^\infty \phi_n(t,\mathbf{x})u_n(z)$$

where $\phi_n \in C^0(\mathbb{R}_t; L^2(\mathbb{R}^3_{\mathbf{x}})) \cap C^1(\mathbb{R}_t; H^{-1}(\mathbb{R}^3_{\mathbf{x}}))$ is solution of the Klein-Gordon equation

(IV.18) $$\partial_t^2\phi_n - \Delta_{\mathbf{x}}\phi_n + \lambda_n^2\phi_n = 0,$$

and the limit (IV.17) holds in $C^0\left(\mathbb{R}_t; L^2(\mathbb{R}^3_{\mathbf{x}}\times\,]0,1[_z)\right) \cap C^1\left(\mathbb{R}_t; W^{-1}\right)$. Moreover we have

(IV.19) $$\|\Phi_0\|_{L^2}^2 + \|\Phi_1\|_{W^{-1}}^2 = \sum_0^\infty \|\phi_n(t)\|_{L^2}^2 + \|(-\Delta_{\mathbf{x}} + \lambda_n^2)^{-\frac{1}{2}}\partial_t\phi_n(t)\|_{L^2}^2.$$

When Φ is a finite energy solution, then $\phi_n \in C^0(\mathbb{R}_t; H^1(\mathbb{R}^3_{\mathbf{x}})) \cap C^1(\mathbb{R}_t; L^2(\mathbb{R}^3_{\mathbf{x}}))$, the limit (IV.17) holds in $C^0\left(\mathbb{R}_t; W^1\right) \cap C^1\left(\mathbb{R}_t; L^2(\mathbb{R}^3_{\mathbf{x}}\times\,]0,1[_z)\right)$, and we have

(IV.20) $$\|\Phi_0\|_{W^1}^2 + \|\Phi_1\|_{L^2}^2 = \sum_0^\infty \|\nabla_{t,\mathbf{x}}\phi_n(t)\|_{L^2}^2 + \lambda_n^2\|\phi_n(t)\|_{L^2}^2.$$

The sequences λ_n and u_n are explicitly given by the formulas (IV.22) and (IV.23) below. In particular, in the case of the gravitational fluctuations for which $\mu = \frac{15}{4}$, λ_n is the set of the strictly positive zeros of the Bessel function $J_1(x)$ and $u_n(z) = \sqrt{2z}\frac{J_2(\lambda_n z)}{J_2(\lambda_n)}$.

First we develop the spectral analysis of the one-dimensional operator

(IV.21)
$$\mathbf{h} := -\frac{d^2}{dz^2} + \frac{\mu}{z^2}, \quad Dom(\mathbf{h}) := \left\{ u \in H^1(]0,1[); \ -u'' + \frac{\mu}{z^2}u \in L^2(]0,1[), \ u(0) = 0, \ u'(1) + \frac{3}{2}u(1) = 0 \right\}.$$

Lemma IV.4. *For all $\mu > -\frac{1}{4}$, $\mathbf{h}$ is a strictly positive selfadjoint operator on $L^2(]0,1[)$ with a compact resolvent. Its spectrum is formed by the sequence of simple eigenvalues $(\lambda_n^2)_{n\in\mathbb{N}}$ defined by*

(IV.22)
$$\lambda_n > 0, \quad J_{\lambda-1}(\lambda_n) = 0, \quad \lambda := \sqrt{\frac{1}{4} + \mu},$$

associated with the L^2-normalized eigenfunctions

(IV.23)
$$u_n(z) := C_n\sqrt{\lambda_n z}J_\lambda(\lambda_n z), \quad C_n := \sqrt{\frac{2\lambda_n}{(4 + \lambda_n^2 - \lambda^2)J_\lambda^2(\lambda_n)}}.$$

Proof of Lemma IV.4. We introduce the set

$$H^1_{(0)}(]0,1[) := \left\{ u \in H^1(]0,1[); \ u(0) = 0 \right\}$$

that is a Hilbert space for the norm $\| u \|_{H^1_{(0)}} := \| u' \|_{L^2}$ which is equivalent to the usual H^1 norm since when $u(0) = 0$ we have

$$\int_0^1 u^2(z)dz \leq \int_0^1 \frac{1}{z^2}u^2(z)dz \leq 4\int_0^1 | u'(z) |^2 \, dz$$

For any $u \in Dom(\mathbf{h})$, $v \in H^1_{(0)}(]0,1[)$, an integration by part gives

$$< \mathbf{h}u, v >_{L^2} = q(u,v) := \frac{3}{2}u(1)v(1) + \int_0^1 u'(z)v'(z) + \frac{\mu}{z^2}u(z)v(z)dz.$$

Since the Hardy inequality shows that

$$q(u,u) \geq \max\left(\left(\mu + \frac{1}{4}\right)\int_0^1 \frac{1}{z^2} | u(z) |^2 \, dz, \ \min(1, 1 + 4\mu)\int_0^1 | u'(z) |^2 \, dz\right),$$

we can see that $\mathbf{h}$ is a symmetric and strictly positive operator on $L^2(0,1)$. Given $f \in L^2(0,1)$, $u_\pm \in Dom(\mathbf{h})$ is solution of $\mathbf{h}u_\pm \pm iu_\pm$ if and only if for all $v \in H^1_{(0)}(]0,1[)$ we have $a_\pm(u_\pm, v) := q(u_\pm, v) \pm i < u_\pm, v >_{L^2} = < f, v >_{L^2}$. The previous inequalities imply that $a_\pm$ is continuous and coercive on $H^1_{(0)}$ hence the Lax-Milgram lemma assures that $u_\pm$ exists, and we conclude that $\mathbf{h}$ is a selfadjoint operator on $L^2(0,1)$, with a compact resolvent by the compact embedding $H^1(]0,1[) \subset\subset L^2(0,1)$. According to the Hilbert-Schmidt theorem, there exists a Hilbert basis of L^2 formed by eigenfunctions $(u_n)_{n\in\mathbb{N}}$ associated to a sequence of eigenvalues $\lambda_n^2 > 0$. The solutions of $-u_n'' + \frac{\mu}{z^2}u_n = \lambda_n^2 u_n$ are given by

$$u_n(z) = \sqrt{\lambda_n z}[\alpha_n J_\lambda(\lambda_n z) + \beta_n Y_\lambda(\lambda_n z)], \quad \lambda := \sqrt{\mu + \frac{1}{4}}, \quad \alpha_n, \beta_n \in \mathbb{C}.$$

From the asymptotics , $J_\lambda(x) \sim x^\lambda$, $Y_\lambda(x) \sim x^{-\lambda}$, $J_\lambda'(x) \sim x^{\lambda-1}$, $Y_\lambda'(x) \sim x^{-\lambda-1}$ as $x \to 0^+$, we deduce that $\beta_n = 0$ when $u_n \in H^1_{(0)}(]0,1[)$. To calculate the constant of normalization, we use the

formula (11.4.5) of [1] with $a = 2$ and $b = 1$ to get

$$\int_0^1 \lambda_n z J_\lambda^2(\lambda_n z) dz = \frac{1}{2\lambda_n}(4 + \lambda_n^2 - \lambda^2)J_\lambda^2(\lambda_n).$$

Finally the condition at $z = 1$ implies $2J_\lambda(\lambda_n) + \lambda_n J_\lambda'(\lambda_n) = 0$ that is equivalent to (IV.22). In particular $J_\lambda(\lambda_n) \neq 0$ and C_n is well defined.

$$Q.E.D.$$

Proof of Theorem IV.3. We use the Hilbert basis $(u_n)_{n\in\mathbb{N}}$ of the previous Lemma. Given $\Phi \in C^0\left(\mathbb{R}_t; L^2(\mathbb{R}_x^3 \times]0, 1[_z)\right)$ we have for all $t \in \mathbb{R}$ and almost all $x \in \mathbb{R}^3$,

$$\Phi(t, x, z) = \lim_{N\to\infty} \sum_{n=0}^N \phi_n(t, x)u_n(z) \ in \ L^2\left(]0, 1[_z, dz\right), \ \ \phi_n(t, x) := \int_0^1 \Phi(t, x, z)u_n(z)dz.$$

The Fubini theorem implies

(IV.24) $$\| \Phi(t, .) \|^2_{L^2(\mathbb{R}_x^3 \times]0,1[_z)} = \sum_{n=0}^\infty \| \phi_n(t, .) \|^2_{L^2(\mathbb{R}_x^3)},$$

and also that

$$\| \phi_n(t, .) - \phi_n(s, .) \|_{L^2(\mathbb{R}_x^3)} \leq \| \Phi(t, .) - \Phi(s, .) \|_{L^2(\mathbb{R}_x^3 \times]0,1[_z)}$$

hence $\phi_n \in C^0\left(\mathbb{R}_t; L^2\left(\mathbb{R}_x^3\right)\right)$. Moreover

$$\| \Phi(t, .) - \sum_{n=0}^N \phi_n(t, .)u_n \|^2_{L^2(\mathbb{R}_x^3 \times]0,1[_z)} = \int_{\mathbb{R}^3} \| \Phi(t, x, .) - \sum_{n=0}^N \phi_n(t, x)u_n(.) \|^2_{L^2(]0,1[_z)} \ dx,$$

and since $\| \Phi(t, x, .) - \sum_{n=0}^N \phi_n(t, x)u_n(z) \|^2_{L^2(]0,1[_z)} \leq 4 \| \Phi(t, x, .) \|^2_{L^2(]0,1[_z)}$, we deduce from the dominated convergence theorem that

$$\Phi(t, x, z) = \lim_{N\to\infty} \sum_{n=0}^N \phi_n(t, x)u_n(z) \ in \ L^2\left(\mathbb{R}_x^3 \times]0, 1[_z\right)$$

and this limit is uniform on the compacts of $L^2\left(\mathbb{R}_x^3 \times]0, 1[_z\right)$. We conclude that for any $\Phi \in C^0\left(\mathbb{R}_t; L^2\left(\mathbb{R}_x^3 \times]0, 1[_z\right)\right)$ we have

(IV.25) $$\Phi(t, x, z) = \lim_{N\to\infty} \sum_{n=0}^N \phi_n(t, x)u_n(z) \ in \ C^0\left(\mathbb{R}_t; L^2\left(\mathbb{R}_x^3 \times]0, 1[_z\right)\right).$$

We can extend L into an isometry $\tilde{L}$ from W^1 onto W^{-1}. Given $\varphi, \psi \in C_0^\infty\left(\mathbb{R}_x^3\right)$ and u_n, u_p, we have

$$\langle \varphi \otimes u_n; \psi \otimes u_p \rangle_{W^{-1}} = \left\langle \tilde{L}^{-1}\left[\varphi \otimes u_n\right]; \tilde{L}^{-1}\left[\psi \otimes u_p\right] \right\rangle_{W^1} = \left\langle \tilde{L}^{-1}\left[\varphi \otimes u_n\right]; \psi \otimes u_p \right\rangle_{L^2}.$$

Since we can check that $L\left(\left[-\Delta_x + \lambda_n^2\right]^{-1}\varphi \otimes u_n\right) = \varphi \otimes u_n$, we deduce that

$$\langle \varphi \otimes u_n; \psi \otimes u_p \rangle_{W^{-1}} = \left\langle \left[-\Delta_x + \lambda_n^2\right]^{-1}\varphi \otimes u_n; \psi \otimes u_p \right\rangle_{L^2(\mathbb{R}_x^3 \times]0,1[_z)}$$

$$= \left\langle \left[-\Delta_x + \lambda_n^2\right]^{-\frac{1}{2}}\varphi; \left[-\Delta_x + \lambda_p^2\right]^{-\frac{1}{2}}\psi \right\rangle_{L^2(\mathbb{R}_x^3)} \delta_{n,p},$$

where $\delta_{n,p}$ is the symbol of Kronecker. We deduce that

$$\| \Phi(t, .) \|^2_{W^{-1}} = \sum_{n=0}^\infty \| \left[-\Delta_x + \lambda_n^2\right]^{-\frac{1}{2}}\phi_n(t, .) \|^2_{L^2}.$$

As a consequence of the density of L^2 in W^{-1}, given $\Phi' \in W^{-1}$, the map

$$\phi'_n : \; \varphi \in H^1(\mathbb{R}^3_x) \mapsto \langle \Phi'; \varphi \otimes u_n \rangle_{W^{-1},W^1}$$

is a well defined distribution of $H^{-1}(\mathbb{R}^3_x)$ and

$$\Phi' = \lim_{N \to \infty} \sum_{n=0}^{N} \phi'_n \otimes u_n \;\; in \;\; W^{-1},$$

(IV.26)
$$\| \Phi' \|^2_{W^{-1}} = \sum_{n=0}^{\infty} \| \left[-\Delta_x + \lambda_n^2 \right]^{-\frac{1}{2}} \phi'_n \|^2_{L^2} .$$

More generally, for any $\Phi' \in C^0 \left(\mathbb{R}_t; W^{-1} \right)$ we have

(IV.27)
$$\langle \phi'_n(t); \varphi \rangle_{H^{-1},H^1} := \langle \Phi'(t); \varphi \otimes u_n \rangle_{W^{-1},W^1} \in C^0 \left(\mathbb{R}_t \right),$$

hence $\phi'_n \in C^0 \left(\mathbb{R}_t; H^{-1} \left(\mathbb{R}^3_x \right) \right)$, and

(IV.28)
$$\Phi'(t) = \lim_{N \to \infty} \sum_{n=0}^{N} \phi'_n(t) \otimes u_n \;\; in \;\; C^0 \left(\mathbb{R}_t; W^{-1} \right).$$

Similarly we have

$$\langle \varphi \otimes u_n; \psi \otimes u_p \rangle_{W^1} = \langle L \left[\varphi \otimes u_n \right]; \psi \otimes u_p \rangle_{L^2} = \langle \nabla_x \varphi; \nabla_x \psi \rangle_{L^2} \delta_n^p + \lambda_n^2 \langle \varphi; \psi \rangle_{L^2} \delta_n^p,$$

hence when $\Phi \in C^0 \left(\mathbb{R}_t; W^1 \right)$ we have

(IV.29)
$$\Phi(t, x, z) = \lim_{N \to \infty} \sum_{n=0}^{N} \phi_n(t, x) u_n(z) \;\; in \;\; C^0 \left(\mathbb{R}_t; W^1 \right),$$

(IV.30)
$$\|\Phi(t)\|^2_{W^1} = \sum_{0}^{\infty} \|\nabla_x \phi_n(t)\|^2_{L^2} + \lambda_n^2 \|\phi_n(t)\|^2_{L^2}.$$

Now we consider a normalizable solution Φ. To prove that ϕ_n defined as above is solution of the Klein-Gordon equation on the Minkowski space-time with the mass λ_n^2, we take $\chi \in C_0^\infty \left(\mathbb{R}_t \times \mathbb{R}^3_x \right)$ and $\theta \in C_0^\infty \left( ]0, 1[_z \right)$, and we write :

(IV.31)
$$
\begin{aligned}
0 &= \left\langle \left(\partial_t^2 - \Delta_x - \partial_z^2 + \frac{\mu}{z^2} \right) \Phi; \chi \otimes \theta \right\rangle_{\mathcal{D}',\mathcal{D}} \\
&= \left\langle \Phi; \left(\partial_t^2 - \Delta_x \right) \chi \otimes \theta + \chi \otimes \left(-\partial_z^2 + \frac{\mu}{z^2} \right) \theta \right\rangle_{L^2} \\
&= \sum_{n=0}^{\infty} \left\langle \phi_n; \left(\partial_t^2 - \Delta_x \right) \chi \right\rangle_{L^2} \langle u_n; \theta \rangle_{L^2} + \langle \phi_n; \chi \rangle_{L^2} \left\langle u_n; \left(-\partial_z^2 + \frac{\mu}{z^2} \right) \theta \right\rangle_{L^2} \\
&= \sum_{n=0}^{\infty} \left\langle \partial_t^2 \phi_n - \Delta_x \phi_n + \lambda_n^2 \phi_n; \chi \right\rangle_{\mathcal{D}',\mathcal{D}} \langle \theta; u_n \rangle_{L^2},
\end{aligned}
$$

and we conclude that ϕ_n is solution of the Klein-Gordon equation (IV.18). Now the expansion (IV.17) is a consequence of (IV.25) and (IV.28). (IV.24) and (IV.26) imply the conservation law (IV.19). When Φ is a finite energy solution, (IV.29) and (IV.25) assure the convergence of (IV.17) in $C^0 \left(\mathbb{R}_t; W^1 \right) \cap C^1 \left(\mathbb{R}_t; L^2(\mathbb{R}^3_x \times]0, 1[_z) \right)$, and the expansion of the energy (IV.20) is given by (IV.24) and (IV.30).

$$Q.E.D.$$

We end this part by a result of uniform decay that is a very modest step toward the much more difficult study of the non-linear stability of the brane. By the way, the $L^2 - L^\infty$ estimates are very useful to the analysis of non-linear problems. We shall use a dyadic partition of the unity $\chi_p \in C_0^\infty([0,\infty[)$ satisfying

$$\sum_{p=0}^\infty \chi_p = 1, \ \ \text{supp}\chi_0 \subset [0,2[, \ \ 1 \le p \Rightarrow \text{supp}\chi_p \subset [2^{p-1}, 2^{p+1}].$$

Given an integer $k \ge 0$, and two integer-valued functions k', k'' defined on $\mathbb{N}^3$, we introduce the functional space

$$\Theta(k, k', k'') := \left\{ \Phi \in L^2\left(\mathbb{R}_{\text{x}}^3 \times]0, 1[_z\right) ; \ | \alpha | \le k \Rightarrow \partial_{\text{x}}^\alpha \Phi \in Dom\left(L^{\max(k'(\alpha), k''(\alpha))}\right), \ \| \Phi \|_{\Theta(k,k',k'')} < \infty \right\},$$

where L is the operator defined by (IV.14). On this space we define the norm :

$$\| \Phi \|_{\Theta(k,k',k'')} := \sum_{|\alpha| \le k} \| (1 + | \text{x} |)^{\frac{3}{2}} \partial_{\text{x}}^\alpha \left(-\partial_z^2 + \frac{\mu}{z^2}\right)^{k'(\alpha)} \Phi \|_{L^2(\mathbb{R}_{\text{x}}^3 \times]0,1[_z)}$$

$$+ \sum_{|\alpha| \le k} \sum_{p=0}^\infty \| \chi_p(| \text{x} |)(1 + | \text{x} |)^{\frac{3}{2}} \partial_{\text{x}}^\alpha \left(-\partial_z^2 + \frac{\mu}{z^2}\right)^{k''(\alpha)} \Phi \|_{L^2(\mathbb{R}_{\text{x}}^3 \times]0,1[_z)}.$$

Theorem IV.5. *There exists $C > 0$ such that for all $\Phi_0 \in \Theta(3, k', k_0'')$ and $\Phi_1 \in \Theta(2, k', k_1'')$, the finite energy solution Φ satisfies the following estimates :*

(IV.32) $$| \Phi(t, \text{x}, z) | \le C(| t | + | \text{x} |)^{-\frac{3}{2}} z^{\lambda + \frac{1}{2}} \left(\| \Phi_0 \|_{\Theta(3,k',k_0'')} + \| \Phi_1 \|_{\Theta(2,k',k_1'')} \right),$$

with

(IV.33) $$k'(\alpha) = \left[\frac{\lambda + 1}{2}\right] + 1, \ \ k_j''(\alpha) = \left[\frac{2\lambda + 5 - 2 | \alpha | - 2j}{4}\right] + 1, \ \ \lambda = \sqrt{\mu + \frac{1}{4}}$$

and

(IV.34) $$| \Phi(t, \text{x}, z) | \le C(| t | + | \text{x} |)^{-\frac{3}{2}} \left(\| \Phi_0 \|_{\Theta(3,k',k_0'')} + \| \Phi_1 \|_{\Theta(2,k',k_1'')} \right),$$

with

(IV.35) $$k'(\alpha) = 1, \ \ k_j''(\alpha) = \left[\frac{2 - | \alpha | - j}{2}\right] + 1.$$

Proof of Theorem IV.5. In a first time, we assume that $\Phi_j \in C^\infty\left(\mathbb{R}_{\text{x}}^3; L^2]0, 1[_z\right)$ and for all $\text{x} \in \mathbb{R}^3$, the maps $z \mapsto \partial_{\text{x}}^\alpha \Phi_j(\text{x}, z)$ belong to $Dom\left(\text{h}^{\max(k'(\alpha), k_j''(\alpha))}\right)$. We estimate $\Phi(t, \text{x}, z)$ by using the expansion (IV.17) :

$$| \Phi(t, \text{x}, z) | \le \sum_{n=0}^\infty | \phi_n(t, \text{x}) | | u_n(z) | .$$

To control $u_n(z)$ we need the asymptotics for the Bessel function and its zeros ([22] p.238, p.247) :

$$J_\lambda(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \lambda\frac{\pi}{2} - \frac{\pi}{4}\right) + O\left(z^{-\frac{3}{2}}\right),$$

$$\lambda_n = \pi\left(n + \frac{\lambda}{2} - \frac{3}{4}\right) + O\left(n^{-1}\right),$$

therefore $C_n \sim \sqrt{\pi}$ and since $J_\lambda(x) \sim x^\lambda$ as $x \to 0$, we deduce that there exists C independent of n such that

$$| u_n(z) | \le C z^{\lambda + \frac{1}{2}} \lambda_n^{\lambda + \frac{1}{2}}.$$

We get :

$$\mid \Phi(t,\mathrm{x},z)\mid\leq Cz^{\lambda+\frac{1}{2}}\sum_{n=0}^{\infty}\lambda_n^{\lambda+\frac{1}{2}}\mid\phi_n(t,\mathrm{x})\mid.$$

We estimate the Klein-Gordon fields by the Lemma 4.3 of [5] that yields to :

$$\mid\phi_n(t,\mathrm{x})\mid\leq C(1+\mid t\mid+\mid\mathrm{x}\mid)^{-\frac{3}{2}}\sum_{|\alpha|+j\leq3}\parallel(1+\mid\mathrm{y}\mid)^{\frac{3}{2}}\partial_{\mathrm{y}}^{\alpha}\phi_n^{j}\parallel_{L^2(\mathbb{R}_{\mathrm{y}}^3)}$$

(IV.36)

$$+C(\mid t\mid+\mid\mathrm{x}\mid)^{-\frac{3}{2}}\sum_{|\alpha|+j\leq3}\sum_{p=0}^{\infty}\lambda_n^{\frac{3}{2}-|\alpha|-j}\|\chi_p(\mid\mathrm{y}\mid)(1+\mid\mathrm{y}\mid)^{\frac{3}{2}}\partial_{\mathrm{y}}^{\alpha}\phi_n^{j}\|_{L^2(\mathbb{R}_{\mathrm{y}}^3)}$$

where the initial data are given by

$$\phi_n^{j}(\mathrm{x}):=\int_0^1\Phi_j(\mathrm{x},z)\overline{u_n(z)}dz.$$

Since for all $\mathrm{x}\in\mathbb{R}^3$ the maps $z\mapsto\partial_{\mathrm{x}}^{\alpha}\Phi_j(\mathrm{x},z)$ belong to $Dom\left(\mathbf{h}^k\right)$ for some k, we have :

$$\phi_n^{j}(\mathrm{x}):=\lambda_n^{-2k}\int_0^1\left(-\partial_z+\frac{\mu}{z^2}\right)^k\Phi_j(\mathrm{x},z)\overline{u_n(z)}dz.$$

Since u_n is a Hilbert basis of $L^2(]0,1[_z)$ we can write

$$\parallel(1+\mid\mathrm{y}\mid)^{\frac{3}{2}}\partial_{\mathrm{y}}^{\alpha}\phi_n^{j}\parallel_{L^2(\mathbb{R}_{\mathrm{y}}^3)}\leq\lambda_n^{-2k}A_n(\alpha,j,k),$$

$$\sum_{n=0}^{\infty}A_n^2(\alpha,j,k)\leq\parallel(1+\mid\mathrm{y}\mid)^{\frac{3}{2}}\partial_{\mathrm{y}}^{\alpha}\left(-\partial_z+\frac{\mu}{z^2}\right)^k\Phi_j\parallel_{L^2(\mathbb{R}_{\mathrm{y}}^3\times]0,1[_z)}^2,$$

$$\parallel\chi_p(\mid\mathrm{y}\mid)(1+\mid\mathrm{y}\mid)^{\frac{3}{2}}\partial_{\mathrm{y}}^{\alpha}\phi_n^{j}\parallel_{L^2(\mathbb{R}_{\mathrm{y}}^3)}\leq\lambda_n^{-2k}B_n(\alpha,j,k,p),$$

$$\sum_{n=0}^{\infty}B_n^2(\alpha,j,k,p)\leq\parallel\chi_p(\mid\mathrm{y}\mid)(1+\mid\mathrm{y}\mid)^{\frac{3}{2}}\partial_{\mathrm{y}}^{\alpha}\left(-\partial_z+\frac{\mu}{z^2}\right)^k\Phi_j\parallel_{L^2(\mathbb{R}_{\mathrm{y}}^3\times]0,1[_z)}^2.$$

Now we apply the Cauchy-Schwartz inequality and we estimate Φ as follows :

$$\mid\Phi(t,\mathrm{x},z)\mid\leq C(\mid t\mid+\mid\mathrm{x}\mid)^{-\frac{3}{2}}z^{\lambda+\frac{1}{2}}\left[\sum_{|\alpha|+j\leq3}\left(\sum_{n=0}^{\infty}\lambda_n^{2\lambda+1-4k}\right)^{\frac{1}{2}}\parallel(1+\mid\mathrm{y}\mid)^{\frac{3}{2}}\partial_{\mathrm{y}}^{\alpha}\left(-\partial_z+\frac{\mu}{z^2}\right)^k\Phi_j\parallel_{L^2(\mathbb{R}_{\mathrm{y}}^3\times]0,1[_z)}\right.$$

$$\left.+\sum_{|\alpha|+j\leq3}\sum_{p=0}^{\infty}\left(\sum_{n=0}^{\infty}\lambda_n^{2\lambda+4-2|\alpha|-2j-4k}\right)^{\frac{1}{2}}\parallel\chi_p(\mid\mathrm{y}\mid)(1+\mid\mathrm{y}\mid)^{\frac{3}{2}}\partial_{\mathrm{y}}^{\alpha}\left(-\partial_z+\frac{\mu}{z^2}\right)^k\Phi_j\parallel_{L^2(\mathbb{R}_{\mathrm{y}}^3\times]0,1[_z)}\right],$$

where the integers k depend on α and j. To be sure the series are convergent, it is sufficient to take $k=\left[\frac{\lambda+1}{2}\right]+1$ in the first sum and $k=\left[\frac{2\lambda+5-2|\alpha|-2j}{4}\right]+1$ in the second one, so we get (IV.32) with (IV.33).

By the same way, since $\sqrt{x}J_{\lambda}(x)\in L^{\infty}(]0,\infty[)$, we have $\sup_n\parallel u_n\parallel_{L^{\infty}(]0,1[)}<\infty$ and

$$\mid\Phi(t,\mathrm{x},z)\mid\leq C\sum_{n=0}^{\infty}\mid\phi_n(t,\mathrm{x})\mid.$$

As previous we obtain

$$\mid\Phi(t,\mathrm{x},z)\mid\leq C(\mid t\mid+\mid\mathrm{x}\mid)^{-\frac{3}{2}}\left[\sum_{|\alpha|+j\leq3}\left(\sum_{n=0}^{\infty}\lambda_n^{-4k}\right)^{\frac{1}{2}}\parallel(1+\mid\mathrm{y}\mid)^{\frac{3}{2}}\partial_{\mathrm{y}}^{\alpha}\left(-\partial_z+\frac{\mu}{z^2}\right)^k\Phi_j\parallel_{L^2(\mathbb{R}_{\mathrm{y}}^3\times]0,1[_z)}\right.$$

$$\left.+\sum_{|\alpha|+j\leq3}\sum_{p=0}^{\infty}\left(\sum_{n=0}^{\infty}\lambda_n^{3-2|\alpha|-2j-4k}\right)^{\frac{1}{2}}\parallel\chi_p(\mid\mathrm{y}\mid)(1+\mid\mathrm{y}\mid)^{\frac{3}{2}}\partial_{\mathrm{y}}^{\alpha}\left(-\partial_z+\frac{\mu}{z^2}\right)^k\Phi_j\parallel_{L^2(\mathbb{R}_{\mathrm{y}}^3\times]0,1[_z)}\right],$$

and we choose $k = 1$ in the first sum and $k = \left[\frac{2-|\alpha|-j}{2}\right] + 1$ in the second one, that yields to (IV.34) with (IV.35).

To achieve the proof of the theorem, we use a procedure of regularization by taking a sequence $\theta_n \subset C_0^\infty(\mathbb{R}_x^3)$, satisfying $0 \le \theta_n$, $| \text{x} | \ge \frac{1}{n} \Rightarrow \theta_n(\text{x}) = 0$, $\int \theta_n(\text{x})d\text{x} = 1$, and given $\Phi \in \Theta(k, k', k'')$, we put $\Phi_n(\text{x}, z) := \int \Phi(\text{x} - \text{y}, z)\theta_n(\text{y})d\text{y}$. It is clear that $\Phi_n \in C^\infty(\mathbb{R}_x^3; L^2]0, 1[_z)$ and since $\partial_x^\alpha \Phi \in Dom\left(L^{\max(k'(\alpha), k''(\alpha))}\right)$, we have $\Phi_n \in C^\infty\left(\mathbb{R}_x^3; Dom\left(L^{\sup(k', k'')}\right)\right)$. That implies that for all $\text{x} \in \mathbb{R}^3$ and all $\alpha \in \mathbb{N}^3$, the maps $z \mapsto \partial_x^\alpha \Phi_n(\text{x}, z)$ belong to $Dom\left(\mathbf{h}^{\max(k'(\alpha), k_j''(\alpha))}\right)$. To end the proof, we show that Φ_n belongs to in $\Theta(k, k', k'')$ and tends to Φ as $n \to \infty$. This a straight consequence of the following properties that are easy to be proved. If $(1+ | \text{x} |)^{\frac{3}{2}}f \in L^2(\mathbb{R}_x^3)$ we have

$$(1+ | \text{x} |)^{\frac{3}{2}} |f * \theta_n| \le C\left[(1+ | \text{x} |)^{\frac{3}{2}} | f |\right] * \theta_n \in L^2\left(\mathbb{R}_x^3\right),$$

$$\| (1+ | \text{x} |)^{\frac{3}{2}} (f * \theta_n - f) \|_{L^2} \le C \| \left[(1+ | \text{x} |)^{\frac{3}{2}}f\right] * \theta_n - (1+ | \text{x} |)^{\frac{3}{2}}f \|_{L^2} + O\left(\frac{1}{n}\right) \xrightarrow[n\to\infty]{} 0,$$

$$\| \chi_p(\text{x})(1+ | \text{x} |)^{\frac{3}{2}} [f * \theta_n] \|_{L^2} \le C \sum_{j=-1}^{1} \| \chi_{p+j}(\text{x})(1+ | \text{x} |)^{\frac{3}{2}}f \|_{L^2},$$

and if $\sum_{p=0}^\infty \| \chi_p(\text{x})(1+ | \text{x} |)^{\frac{3}{2}}f \|_{L^2} < \infty$, then $\sum_{p=0}^\infty \| \chi_p(\text{x})(1+ | \text{x} |)^{\frac{3}{2}} [f * \theta_n] \|_{L^2} < \infty$ and

$$\sum_{p=0}^{\infty} \| \chi_p(\text{x})(1+ | \text{x} |)^{\frac{3}{2}} (f * \theta_n - f) \|_{L^2} \xrightarrow[n\to\infty]{} 0.$$

$$Q.E.D.$$

V. APPENDIX

The Anti-de-Sitter space in $4 + 1$ dimensions of constant curvature $-k < 0$ can be represented as the quadric

$$X_0^2 + X_5^2 - \sum_{i=1}^{4} X_i^2 = \frac{1}{k^2},$$

embedded in the 4+2 dimensional flat space $\mathbb{R}_X^6$ with metric

$$ds^2 = dX_0^2 + dX_5^2 - \sum_{i=1}^{4} X_i^2.$$

This metric is a maximally symmetric solution without singularity, of the Einstein equations in the vacuum, with the negative cosmological constant $-k < 0$, moreover this solution is unique, for the given topology, according to the Birkhoff theorem.

We can describe this manifold with a unique chart by using the so-called global coordinates τ, ρ, Ω defined by the relations

$$X_0 = \frac{1}{k} \sec \rho \cos \tau, \quad X_5 = \frac{1}{k} \sec \rho \sin \tau, \quad X_i = \frac{1}{k} \tan \rho\, \Omega_i, \quad i = 1, .., 4, \quad \sum_{i=1}^{4} \Omega_i^2 = 1.$$

Then the metric has the form

$$ds^2 = \frac{1}{(k \cos \rho)^2} \left[d\tau^2 - d\rho^2 - (\sin \rho)^2 d\Omega_{S^3}^2\right], \quad \tau \in [-\pi, \pi], \quad \rho \in [0, \frac{\pi}{2}[, \quad \Omega \in S^3.$$

Here, the points at $\tau = \pi$ and $\tau = -\pi$ have to be identified therefore this universe is totally vicious as regards the causality, and we prefer to work with $\tau \in \mathbb{R}$ that corresponds to the universal covering $CAdS^5$. Whatever choice for τ, there is a time-like boundary at $\rho = \frac{\pi}{2}$, the "horizon".

In the coordinates, the d'Alembertian

$$\Box_g := \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu}\left(\sqrt{|g|}\,g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right)$$

is expressed as :

$$\Box_g = (k\cos\rho)^2\left[\partial_\tau^2 - \partial_\rho^2 - \frac{3}{\cos\rho\sin\rho}\partial_\rho - \frac{1}{\sin^2\rho}\Delta_{S^3}\right].$$

To cancel the first derivative with respect to ρ we make a change of unknown and the Klein-Gordon equation becomes :

$$\left(\Box_g + \lambda k^2\right)u = (k\cos\rho)^2(\tan\rho)^{-\frac{3}{2}}\left[\partial_\tau^2 - \partial_\rho^2 + \frac{3}{4\sin^2\rho} + \frac{\frac{15}{4}+\lambda}{\cos^2\rho} - \frac{1}{\sin^2\rho}\Delta_{S^3}\right]\left((\tan\rho)^{\frac{3}{2}}u\right).$$

Following [15], formulae (56), (79), (90), the vector type electromagnetic fields and the vector type gravitational fluctuations obey to this equation with $\lambda = -3$, and the tensor type gravitational fluctuations are solutions with $\lambda = 0$.

The Poincaré patch $\mathcal{P}$ is the domain $X_0 > X_4$ of the Anti-de Sitter space. We introduce the Poincaré coordinates t, x, z defined by :

$$z = \frac{1}{k^2(X_0 - X_4)}, \quad t = kzX_5, \quad \mathrm{x}^i = kzX_i, \quad i = 1,2,3.$$

Then the Poincaré chart $\mathcal{P}$ of AdS^5 is simply

$$\mathcal{P} := \mathbb{R}_t \times \mathbb{R}^3_{\mathbf{x}} \times]0,\infty[_z,$$

endowed with the conformally flat metric

$$ds^2 = \left(\frac{1}{kz}\right)^2\left(dt^2 - d\mathbf{x}^2 - dz^2\right).$$

Its time-like horizon $z = 0$ is a part of the global boundary $\rho = \frac{\pi}{2}$ (see [7] for a complete description). The 4-dimensional Minkowski brane with a negative (resp. positive) tension is the boundary $z = 1$ of the bulk $\mathcal{B}_-$ given by $0 < z < 1$ (resp. $\mathcal{B}_+$ given by $1 < z$).
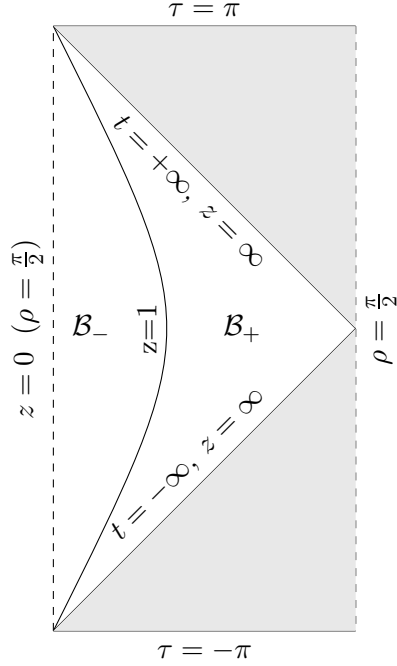
Conformal Penrose diagram of AdS^5. $\mathcal{P}$ is the white domain.
The gray zone is the copy of the Poincaré chart defined by $X_0 < X_4$.

With these nice coordinates, the Klein-Gordon equation is writen as

$$\left[\Box_g + \lambda k^2\right] u = k^2 z^2 \left[\partial_t^2 - \Delta_{\mathbf{x}} - \partial_z^2 + \frac{3}{z}\partial_z + \frac{\lambda}{z^2}\right] u,$$

that we transform to get the free wave equation on a half 5-dimensional Minkowski space-time, pertubed by a singular potential μz^{-2} :

$$\left[\Box_g + \lambda k^2\right] u = k^2 z^{\frac{7}{2}} \left[\partial_t^2 - \Delta_{\mathbf{x}} - \partial_z^2 + \frac{\mu}{z^2}\right]\left(z^{-\frac{3}{2}}u\right), \quad \mu = \frac{15}{4} + \lambda.$$

With Z_2 orbifold symmetry imposed across the brane, the boundary condition for u is Neumann on the brane, $\partial_z u = 0$, hence for $\Phi =: z^{-\frac{3}{2}}u$, this constraint becomes of Robin type :

$$\partial_z \Phi(t, \mathrm{x}, 1) + \frac{3}{2}\Phi(t, \mathrm{x}, 1) = 0.$$

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